The Geometric Mean

I have become all things to all people so that **by all possible means** I might save some.
– 1 Corinthians 9:22

Instructions

- Read everything carefully, and follow all instructions.
- Do the numbered problems that are boxed. Show your work neatly in the allotted space.
- To receive full credit, you must justify your answer by showing your work or calculator commands.
- Circle your final answer, or write it in the spot provided.
- You may work with others, or ask for help. Your answers should reflect your own understanding of the material.
- Select, unspecified, parts of this take-home project may be graded to determine $\boxed{}$ % of your grade.
- Neatly tear the pages out of your book, and have them prepared to submit in class on the due date.
- On the due date, there will be a short in-class portion to determine $\boxed{}$ % of your grade.

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1 Motivating Examples

1.1 ADHD

In the U.S., diagnosis of ADHD doubled between 1993 and 2013.
What was the "average" annual percentage increase?
Considering the 1993 baseline, you might think about the problem this way:

- Doubling corresponds to a 100% increase.
- 20 years elapsed
- Therefore the average change was $\frac{100}{20} = 5$ percent per year.

Sorry, but this answer is incorrect. The reason is that each year's percent change is **compounded** on top of the previous year's total. For illustration, suppose 400,000 were diagnosed in 1993. Let's apply a 5% change each year for 20 years. Each year, multiply the previous year's value by 105% = 1.05.

year	change	factor	value
1993	N/A	N/A	400,000
1994	5%	1.05	420,000
1995	5%	1.05	441,000
1996	5%	1.05	463,050
⋮	"	"	⋮
2013	5%	1.05	1,061,319

Doubling implies 800,000 diagnoses in 2013, but a 5% annual change results in more than a million. Notice that the consecutive application of 5% changes can be computed quickly with an exponent:

$$400,000(1.05)^{20} = 1,061,319$$

Since 5% didn't work, let's use that last equation to find the correct answer. Replace the .05 with an x, and set the right-hand-side to the desired quantity. Then solve for x.

$$\begin{aligned} 400,000(1+x)^{20} &= 800,000 \\ (1+x)^{20} &= 2 \\ 1+x &= 2 \wedge (1/20) \\ 1+x &= 1.0353 \\ x &= .0353 \end{aligned}$$

Therefore, on average, the diagnosis of ADHD increased by 3.53 % per year over the previous year.

1. (4 pts) The price of a 30 second TV ad during the Super Bowl has risen from \$42 thousand in 1967 to \$4 million in 2014. Solve the equation $42,000(1+x)^{47} = 4,000,000$ to find the average annual percent increase in the ad price.
   Answer: $x = 10.2\%$

1.2 Investing

Suppose you invest \$1000 for two years.

- The first year, you gain 40%.
- The second year, you lose 30%.

What is your "average" rate of return on this investment? At first glance, you may be tempted to compute the arithmetic mean of your yearly percent changes.

$$\frac{40 + (-30)}{2} = \frac{10}{2} = 5$$

So it looks like you averaged a 5% per year gain. But what really happened to your money if the returns were compounded? **Multiply** each year's value by a factor to obtain the next year's value.

year	change	factor	value
0	N/A	N/A	1000
1	+40%	$1 + .40 = 1.40$	1400
2	-30%	$1 - .30 = 0.70$	980

After two years, you have actually lost \$ 20, or 2 percent of your original investment. Your true "average" annual return is more like -1 % per year!

2. (4pts) Suppose that over a three year span, a compounded investment was
   - down 50%
   - up 20%
   - up 60%

   Did you make or lose money? What was the overall percent change from the original value.
   Answer: $(0.5)(1.20)(1.60) = 0.96$, so you lost 4% of the original value

3. (3 pts) If you lose 20% one year, then you need to gain $\boxed{\quad}$ percent the following year to get back to even.
   Answer: solve $(0.80)x = 1$ to get $x = 1.25$, so you need to gain 25%

4. (2 pts) The factor 1.083 corresponds to a $\boxed{\quad}$ percent increase.

   Answer: 8.3%

5. (2 pts) The factor 0.803 corresponds to a $\boxed{\quad}$ percent decrease.

   Answer: 19.7%

1.3 Compound Changes

In the previous example, the final value of your investment is

$$1000(1.40)(0.70) = 980$$

Re-arrange to see that the ratio of your final and starting values is the product of the annual factors:

$$\frac{980}{1000} = (1.40)(0.70)$$

When percent changes are **compounded**, the net result is based on repeated multiplication, not addition. That is why the arithmetic mean was misleading. We will need to use a different kind of "average" in situations like this.

Let x be the "average" annual percent change, so that the multiplication factor is $1 + x$ each year. We would like two years of "average" change to equal the net effect of up 40% and then down 30%. Algebraically, we require:

$$(1 + x)^2 = (1.40)(0.70)$$

The annual factor is therefore:

$$\begin{aligned} 1 + x &= \sqrt{(1.40)(0.70)} \\ &= \sqrt{0.98} \\ &= 0.98995 \end{aligned}$$

We will call this number the "**geometric mean**" of 1.40 and 0.70. Subtract one from both sides to get

$$x = 0.98995 - 1 = -.01005 = -1.005\%$$

So as we noticed earlier, our true "average" annual return is approximately a 1% loss.

6. (4 pts) Find the net percentage gain for each of these compounded investments over a four year period. Which did better?

   - Investment A: $+10\%, +12\%, +9\%, +7\%$
   - Investment B: $+10\%, +31\%, -15\%, +12\%$

   Answer: A went up by $(1.10)(1.12)(1.09)(1.07) - 1 = 1.437 - 1 = 0.437 = 43.7\%$
   B went up by $(1.10)(1.31)(0.85)(1.12) - 1 = 1.372 - 1 = 0.372 = 37.2\%$
   so consistent returns do better

7. (4 pts) If a compounded investment returned +10%, +50%, +200% for three consecutive periods, what was the "average" rate of return per period?
   Hint: Solve the equation $(1 + x)^3 = (1 + .10)(1 + .50)(1 + 2.00)$.
   Answer: solve $(1+x)^3 = (1+.10)(1+.50)(1+2.00) = 4.95$ to get $1+x = 1.704$, so $x = 0.704 = 70.4\%$

2 Mean Formulas

Suppose we have a set of data:

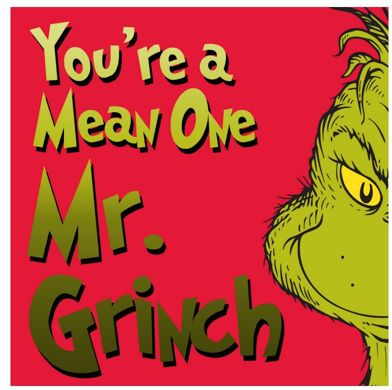
$$\{x_1, x_2, x_3, \cdots, x_n\}$$

2.1 Arithmetic Mean

Recall that the **arithmetic mean** is an average computed as follows:

- add up the values to find the total sum
- divide by the sample size n

As a mathematical formula, we write



$$\overline{x} = \frac{\sum x_i}{n}$$

For example, the arithmetic mean of $\{2, 8\}$ is

$$\overline{x} = \frac{2+8}{2} = \frac{10}{2} = 5$$

2.2 Geometric Mean

The **geometric mean** is an average computed as follows:

- multiply the values to find the product
- take the nth root

As a mathematical formula, we write

$$\begin{aligned} \overline{x}_g &= \sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdots x_n} \\ &= (x_1 \cdot x_2 \cdot x_3 \cdots x_n) \wedge (1/n) \end{aligned}$$

(Note the $\wedge$ symbol represents exponentiation on your calculator.)
For example, the geometric mean of $\{2, 8\}$ is

$$\overline{x}_g = (2 \cdot 8) \wedge (1/2) = (16) \wedge (1/2) = 4$$

The reason this mean is called "geometric" is that a rectangle with sides of length 2 and 8 has the same area as a square with sides of length 4.

2.3 Compare and Contrast

Note the similar pattern for computing these means. The arithmetic mean reflects the sum of the data, while the geometric mean reflects the product.

data	arithmetic mean $\overline{x}$	geometric mean $\overline{x}_g$
{2,8}	$\frac{10}{2} = 5$	$\sqrt{16} = 4$
{1,2,18,36}	$\frac{57}{4} = 14.25$	$\sqrt[4]{1296} = 6$
{10,10,20}	$\frac{40}{3} = 13.33$	$\sqrt[3]{2000} = 12.60$
{0,10,20}	$\frac{30}{3} = 10$	$\sqrt[3]{0} = 0$
{3,3,3,3}	$\frac{12}{4} = 3$	$\sqrt[4]{81} = 3$

We will not use the geometric mean if the data set contains negative numbers. Note that the geometric mean is never larger than the arithmetic mean.

8. (3 pts) Compute the geometric mean of 20 and 45.
Answer: $\overline{x}_g = \sqrt{20 \cdot 45} = 30$

9. (3 pts) Compute the geometric mean of $\{2, 9, 50\}$.
Answer: $\overline{x}_g = (2 \cdot 9 \cdot 50) \wedge (1/3) = 9.65$

10. (3 pts) Compute the geometric mean of $\{1, 2, 3, 4, 5\}$.
Answer: $\overline{x}_g = (120) \wedge (1/5) = 2.605$

11. (3 pts) The numbers $\{2, 7, 12, 20, x\}$ have $\overline{x}_g = 14$. Find the value of x.
Answer: solve $(3360x)^{(1/5)} = 14$ to get $x = 160$

12. (4 pts) Find $\overline{x}$ and $\overline{x}_g$ for each data set. For which situation is the difference $\overline{x} - \overline{x}_g$ smaller?

    - the x values are about the same size, e.g. $\{17, 19, 20, 22, 22\}$
      $\overline{x} = \boxed{}$ and $\overline{x}_g = \boxed{}$, so the difference is $\boxed{}$.
    - the x values are widely dispersed, e.g. $\{2, 9, 20, 27, 42\}$
      $\overline{x} = \boxed{}$ and $\overline{x}_g = \boxed{}$, so the difference is $\boxed{}$.

    Answer: in the first case, $\overline{x} - \overline{x}_g = 20 - 19.91 = .09$
    in the second case, $\overline{x} - \overline{x}_g = 20 - 13.25 = 6.75$
    so $\overline{x}$ and $\overline{x}_g$ are closer when the x values have little variance

3 Examples

3.1 Average Rates of Change

Use the geometric mean, $\overline{x}_g$, whenever the data represent compounded factors or percent changes. To find the "average" compound rate of change, follow this procedure:

- take the geometric mean of the factors
- subtract one
- move the decimal point two places to write the average percent change

For the years 2007-2013, the U.S. stock market changed by 5%, -37%, 27%, 15%, 2%, 16%, 32% respectively. To find the average annual change:

- $\overline{x}_g = ((1.05)(0.63)(1.27)(1.15)(1.02)(1.16)(1.32)) \wedge (1/7) = 1.0605$
- $1.0605 - 1 = 0.0605$
- so the average percent gain was 6.05% per year

You may calculate that the arithmetic mean of the yearly percent changes is $\overline{x} = 8.57$, which is misleadingly high and doesn't reflect what really happened to stock investments.

3.2 Rankings

There are other situations where the geometric mean offers a good alternative to the arithmetic mean. For example, the geometric mean is a good choice for averaging rankings, where low numbers indicate a better outcome.

Consider two NASCAR drivers with listed finishing positions.

- Driver A: $\{1, 2, 5, 8, 17, 39\}$
- Driver B: $\{10, 10, 11, 11, 12, 12\}$

Using the arithmetic mean, we would determine that driver B is slightly better:

- Driver A: $\overline{x} = \frac{72}{6} = 12$
- Driver B: $\overline{x} = \frac{66}{6} = 11$

However any race fan would judge that Driver A's accomplishments are more impressive (1 win and 3 top 5 finishes). The geometric mean naturally emphasizes the lower numbers.

- Driver A: $\overline{x}_g = (1 \cdot 2 \cdot 5 \cdot 8 \cdot 17 \cdot 39)^{1/6} = 6.13$
- Driver B: $\overline{x}_g = (10 \cdot 10 \cdot 11 \cdot 11 \cdot 12 \cdot 12)^{1/6} = 10.97$

13. (3pts) Suppose your salary changed by 40%, −15%, and 20% the last three years respectively. Use the geometric mean to find the average annual increase.
Answer: $((1.40)(0.85)(1.20))^{1/3} = 1.126$, so 12.6% annually

14. (3 pts) Two football teams have been ranked by various polls. Compute the geometric mean of each team's rankings.

    (a) Aardvarks: $\{1, 1, 2, 3, 7\}$
    Answer: $(42)^{1/5} = 2.11$

    (b) Butterflies: $\{2, 2, 3, 4, 2\}$
    Answer: $(96)^{1/5} = 2.49$

15. (3 pts) A car magazine article ranks vehicles in various categories.

	Kia	Honda	Toyota	Ford
comfort	3	2	1	4
fuel economy	1	2	3	4
performance	2	4	1	3
style	4	2	3	1
safety	3	1	4	2
cost	2	4	3	1

Note that each car has the same arithmetic mean ranking.
Compute the geometric mean of each car's rankings:
Kia $\boxed{\quad}$, Honda $\boxed{\quad}$, Toyota $\boxed{\quad}$, Ford $\boxed{\quad}$.
Answer: 2.29, 2.245, 2.18, 2.14

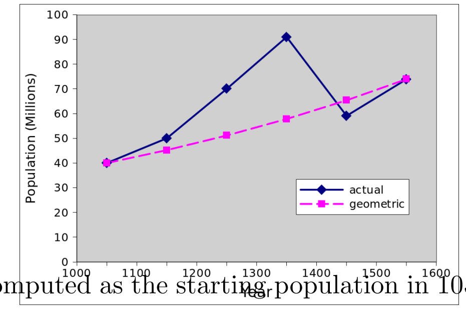
3.3 Population

The population of Western Europe was about 40 million in the year 1050. It changed by roughly

$$\{25, 40, 30, -35, 25\}$$

percent over the next five centuries respectively. This table describes the population change.

year	change	factor	population
1050	N/A	N/A	40
1150	+25%	1.25	50
1250	+40%	1.40	70
1350	+30%	1.30	91
1450	-35%	0.65	59.1
1550	+25%	1.25	73.9



Note that the final population in 1550 can be computed as the starting population in 1050 multiplied by the sequence of factors.

$$40(1.25)(1.40)(1.30)(0.65)(1.25) = 73.9$$

The geometric mean of the factors is:

$$\overline{x}_g = ((1.25)(1.40)(1.30)(0.65)(1.25)) \wedge (1/5) = 1.131$$

Subtracting one, we get $1.131 - 1 = .131 = 13.1\%$, so on average, the population grew by 13.1% per century.

4 Shortcut

Note that the ratio of final (73.9 million) and starting (40 million) populations equals the product of the factors:

$$\frac{73.9}{40} = (1.25)(1.40)(1.30)(0.65)(1.25)$$

Therefore the geometric mean can be computed more easily as

$$\overline{x}_g = \left(\frac{73.9}{40}\right)^{1/5} = 1.131$$

In general, whenever we compute the geometric mean of compounded factors, we do not need to know the intermediate ups and downs; only the starting and final values are required. The average per-period factor is:

$$\overline{x}_g = \left(\frac{\text{final value}}{\text{starting value}}\right)^{(1/n)}$$

where n is the number of periods elapsed.

It is best to enter this formula on one line in your calculator to avoid rounding error. For example, if a quantity increased from 200 to 700 in 9 perods, then enter

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                (700/200)^(1/9)
    Answer:         1.149349219
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To find the "average" percent change under the effects of compounding, you must subtract $1 = 100\%$ from the geometric mean of the factors.

- factor $\overline{x}_g = 1.149$ corresponds to $1.149 - 1 = .149 = 14.9\%$ average increase

- factor $\overline{x}_g = 0.95$ corresponds to a $0.95 - 1 = -0.05 = -5\%$ change, or 5% average decrease

16. (3 pts) Suppose a quantity increased from 30 to 60 over 10 days. Then the average daily factor is $\left(\frac{60}{30}\right)^{1/10} = 1.0718$, which corresponds to an average daily increase of $\boxed{}$ percent.

    Answer: 7.18%

5 Examples

In the remaining examples, we will compute the average compounded percent change using only the starting and final values.

- When the quantity has increased, the per-period factor is greater than one, and we have a positive change, or **growth**.
- When the quantity has decreased, the per-period factor is less than one, and we have negative change, or **decay**.

5.1 Inflation

Consider the time period Jan. 1, 1938 through Jan. 1, 2011. Note that is a 73 year span.

1. During that time, the stock market index rose from 11 to 1280. The “average” annual factor was:

$$\left(\frac{1280}{11}\right)^{1/73} = 1.0673$$

corresponding to a 6.73 % average annual return.

2. The consumer price index (CPI) in 1938 was 14; in 2011 the CPI was 220. Since,

$$\left(\frac{220}{14}\right)^{1/73} = 1.0385$$

there was an average of 3.85% annual inflation during that period.

3. In 2010, a Danbury mint commercial bragged that a “special edition” 1938 coin was now worth 1200% more than face-value. Was this a good investment?

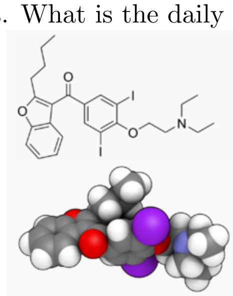
No! If a nickel went up 1200%, then it would be valued at 65 cents, so the average annual return was

$$\left(\frac{65}{5}\right)^{1/73} = 1.0358$$

The coin increased in value 3.58% per year, which is less than inflation.

5.2 Pharmacokinetics

Pharmacokinetics is the study of what happens to a drug inside a body. Suppose the drug amiodarone has a half-life of 36 days. What is the daily decay rate?



Recall from science class that the term “half-life” is the length of time required for one-half of the substance to decay. In this example, 100 mg would decay to 50 mg in 36 days. Therefore the average daily factor is:

$$\left(\frac{50}{100}\right)^{1/36} = 0.981$$

Subtract one to get $0.981 - 1 = -0.019$, so 1.9% is eliminated each day.

17. (3 pts) When Carson-Newman was founded in 1851, the population of Tennessee was about 1 million. In 2014, the population was about 6.5 million. Use the geometric mean to find the average annual percent change in population.
Answer: $\left(\frac{6.5}{1}\right)^{(1/163)} = 1.01155$, so about 1.155 % annually

18. (3 pts) According to the World Bank and OECD, United States per-capita health care expendatures increased from \$2600 in 1990 to \$8200 in 2010. Find the average annual percent increase.
Answer: $\left(\frac{8200}{2600}\right)^{(1/20)} = 1.0591$, then $1.0591 - 1 = .0591 = 5.91\%$ annual increase

19. (3 pts) Because of rampant poaching, the African black rhino population dropped from 65,000 in 1970 to only 3,900 in 2005.

Use the geometric mean to find the average annual percent loss.
Answer: $\left(\frac{3900}{65000}\right)^{(1/35)} = 0.9228$, then $0.9228 - 1 = -0.0772 = -7.72\%$, so a 7.72% annual decrease

20. (3 pts)A newspaper's print circulation is 35% lower than it was ten years ago. Find the average annual percent decline.
Answer: $\left(\frac{65}{100}\right)^{(1/10)} = 0.9578$, then $.9578 - 1 = -.0422$ indicates a 4.22% annual decline

21. (3 pts)In 1964, Warren Buffett purchased Berkshire Hathaway for \$11.50 per share.

Fifty years later, the price has increased to \$ 165,000 per share. Find the average annual compounded rate of return.
Answer: $\left(\frac{165000}{11.5}\right)^{(1/50)} = 1.211$, so about 21.1 % annually

22. (4 pts) A new car was purchased in 2000 for $\$14,500$. It was sold ten years later for $\$3,500$.

(a) Find the average annual depreciation rate.
Answer: $\left(\frac{3500}{14500}\right)^{(1/10)} = .8675$, subtract 1 to get $-.1325$, so it lost an average of 13.25% of its value per year

(b) If that depreciation continued four more years, estimate the car's value in 2014.
Answer: $3500(.8675)^4 = 1982$ dollars

23. (3 pts) Suppose an investment tripled in five years. Find the average annual percent increase.
Answer: $(3)^{(1/5)} = 1.246$, so 24.6% annually

24. (3 pts) The worldwide number of malaria deaths dropped from 835 thousand in 2004 to 627 thousand in 2012.

Find the average annual percent decrease.
Answer: $(627/835)^{(1/8)} - 1 = -.0352$, so a 3.52% annual decrease

25. (4 pts) Some economists have argued that the geometric mean should be used to assess average employee wages. This is because low/middle income employees derive more economic utility from a dollar than a highly paid executive does. Consider two companies, each having four employees with the listed salaries (in thousands of dollars).

- Company A: $\{20, 25, 35, 100\}$
- Company B: $\{30, 35, 45, 60\}$

(a) The arithmetic mean wage for A was \$ $\boxed{}$, and for B it was \$ $\boxed{}$.

Answer: $45, 42.5$

(b) The geometric mean wage for A was \$ $\boxed{}$, and for B it was \$ $\boxed{}$.

Answer: $36.4, 41.0$