# Wed, Mar 18 - Example 1

Let $A \in \mathbb{R}^{4\times 2}$ with $a_{ij} = i - 2j$.

1. Write $A$.

$$A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$$

2. Find $B = A^T A$.

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix}$$

3. Find $(A^T A)^{-1}$. Note that $\det(B) = (6)(14) - (2)(2) = 80$

$$\frac{1}{80} \begin{bmatrix} 14 & -2 \\ -2 & 6 \end{bmatrix}$$

4. Find a LU factorization of $B$. Do one row operation to get $U$.

$$\begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 40/3 \end{bmatrix}$$

Putting the shear on the other side, we get $B = LU$ as:

$$\begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 0 & 40/3 \end{bmatrix}$$

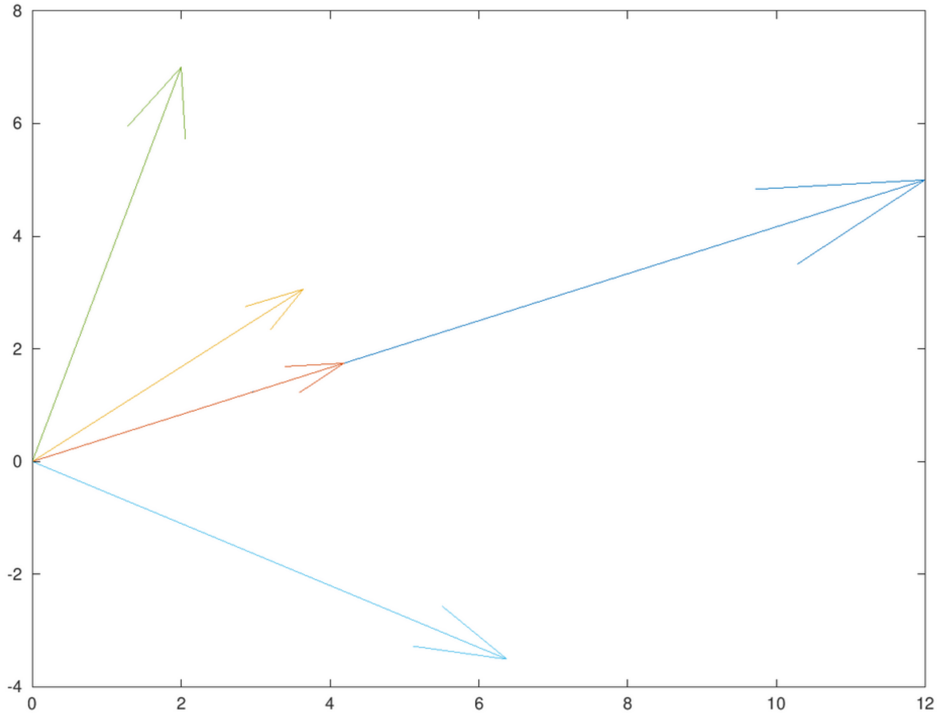
5. Find a QR factorization of $B$. Let $\theta = \tan^{-1} 2/6$, and let $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} .949 & -.316 \\ .316 & .949 \end{bmatrix}$. Then let

$$R = Q^T B = \begin{bmatrix} .949 & -.316 \\ .316 & .949 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 6.325 & 6.325 \\ 0 & 12.65 \end{bmatrix}$$

You can check that now $QR = B$.

## Wed, Mar 18 - Example 2

Let $v = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$ (long dark blue), and $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (green).
You may try this yourself at `https://massey.limfinity.com/208/sage.htm`.




1. Find the angle between $v$ and $x$.

$$\cos\theta = \frac{v^T x}{\|v\|\|x\|} = \frac{59}{\sqrt{169}\sqrt{53}} = .62341$$

$$\theta = \cos^{-1}(.62341) = .89771 = 51.4^\circ$$

2. Find the matrix $P$ that projects onto the line $y = \frac{5}{12}x$. Note that this line is just the extension (span) of the vector $v$.

$$P = \frac{vv^T}{v^T v} = \frac{1}{12^2 + 5^2} \begin{bmatrix} 12 \\ 5 \end{bmatrix} [12\ 5] = \frac{1}{169} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix}$$

3. Project the vector $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ onto $v$. (orange)

$$Px = \frac{1}{169} \begin{bmatrix} 144 & 60 \\ 60 & 60 \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 708/169 \\ 295/169 \end{bmatrix}$$

4. Find the matrix $R$ that reflects across the line $y = \frac{5}{12}x$.

$$2P - I = \frac{2}{169} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{169} \begin{bmatrix} 119 & 120 \\ 120 & -119 \end{bmatrix}$$

5. Reflect the vector $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ across $v$. (light blue)

$$Rx = (2P - I)x = 2Px - x = 2\begin{bmatrix} 708/169 \\ 295/169 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \frac{1}{169}\begin{bmatrix} 1078 \\ -593 \end{bmatrix}$$

6. Form the "flattening" matrix $F(.25) = .75P + .25I$, which essentially does 75% of the projection, leaving vectors 25% as far from $v$ as they started.

$$F(.75) = \frac{3}{4}\frac{1}{169}\begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{169}\begin{bmatrix} 601 & 45 \\ 45 & 61 \end{bmatrix}$$

7. Flatten the vector $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ around $v$. (yellow)

$$F(.75)x = (.75P + .25I)x = .75Px + .25x = .75\begin{bmatrix} 708/169 \\ 295/169 \end{bmatrix} + .25\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1231/338 \\ 517/169 \end{bmatrix}$$