Homework - Spring 2020

Vector, Matrix Notation

1. Let $x = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

   (a) State the dimension of these vectors.

   (b) Find x_2.

   (c) Write the transpose x^T as a row vector.

   (d) Write $(x^T)^T$.

   (e) Compute the dot product $x \cdot y$.

   (f) Compute $y \cdot x$.

   (g) Change the value of y_3 to force $x \cdot y = 0$.

   (h) Consider a vector of ones: $\vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Compute $\vec{1} \cdot x$. What is this accomplishing ?

   (i) Compute the linear combination $5x - 3y$.

   (j) Compute $\vec{1} \cdot (5x - 3y)$. Does it equal $5(\vec{1} \cdot x) - 3(\vec{1} \cdot y)$?

2. Let $A = \begin{bmatrix} 2 & 5 & 1 & 8 & 7 \\ 4 & 9 & 7 & 0 & 2 \\ 0 & 2 & 1 & -6 & 3 \end{bmatrix}$.

   (a) State the $m \times n$ dimensions of this matrix.

   (b) Find a_{23}.

   (c) Write the indicies of the entries with $a_{ij} = 0$.

   (d) Write $a_{:2}$, i.e. the 2nd column.

   (e) Write $a_{3:}$, i.e. the 3rd row.

   (f) Compute $a_{:2} \cdot a_{:5}$

   (g) Compute $a_{2:} \cdot a_{3:}$

   (h) Write the transpose matrix A^T.

   (i) Is it always true that $(A^T)^T = A$?

Linear Functions

3. A worker makes \$ 9 per hour. Let $f(t)$ be the wages earned in t hours. Is $f(t)$ linear ? Compute $f(30)$.

4. Suppose you deposit \$ 1000 in a bank account pays 4% annual interest. You leave the money alone and let it compound. Let $f(t)$ be the total amount of interest earned after t years. Compute $f(1)$ and $f(2)$. Is $f(t)$ linear ?

5. Let $f(x)$ be your profit if you charge x dollars for your fund raiser's candy bars. Is $f(x)$ linear ?

6. A recipe calls for 2/3 cup of sugar to make a dozen cookies. Let $f(x)$ be the amount of sugar needed to make x cookies. Is $f(x)$ linear ?

7. A batch of cookies requires 2/3 cup of sugar, and a batch of brownies requires 2 cups of sugar. If you make b batches of brownies, and c batches of cookies, write your sugar requirement as a linear function $f(b, c)$.

8. An apple costs 75¢, and a banana is 40¢. Let the bill be $f(a, b)$, where a and b are the number of each fruit purchased.

   (a) Xerxes buys 3 apples and 5 bananas; how much is his bill $f(3, 5)$?

   (b) Yolanda buys 4 apples and 8 bananas; how much is her bill $f(4, 8)$?

   (c) Xerxes offers to pay for all the fruit; what is the total bill $f(7, 13)$?

   (d) Is it true that $f(a_1, b_1) + f(a_2, b_2) = f(a_1 + a_2, b_1 + b_2)$?

   (e) Let $k \in \mathbb{N}$ be a constant. Is it true that $f(ka, kb) = kf(a, b)$?

9. You own a portfolio of 100 shares of AAPL, 60 shares of GOOG, and 250 shares of MSFT.

   (a) Write the value of your portfolio a linear function of the current prices per share: $f(a, g, m)$.

   (b) If all three stocks rise by 2% on bullish news for the tech industry, does the value of your portfolio go up by 2% ? i.e. is it true that $f(1.02a, 1.02g, 1.02m) = 1.02f(a, g, m)$?

Linear Combinations

10. Let $x = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

    (a) Find $x \cdot y$, $x \cdot x$, and $y \cdot y$.

    (b) Find $\|x\|$ and $\|y\|$.

    (c) Find $\|-3x\|$. What can you say about $\|cx\|$ for $c \in \mathbb{R}$?

    (d) Find $\|x + y\|$. Is it less than, equal to, or greater than $\|x\| + \|y\|$?

    (e) Find $x \cdot (3x - y)$. Does the distributive property work ?

    (f) Find $(3x - y) \cdot (3x - y)$. Express it in terms of $\|x\|$, $\|y\|$, and $x \cdot y$

    (g) Find $\|3x - y\|$.

    (h) Write $3x - y$ as a matrix-vector multiplication.

11. Let $A = \begin{bmatrix} 5 & 4 & 5 & 7 \\ 9 & 7 & 8 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix}$.

    (a) Write these linear combinations of A's columns as a matrix-vector multiplication, and compute the result.

        i. $2a_2 + 7a_4$

        ii. $5a_2 - 3a_1 - a_3$

    (b) Write these linear combinations of A's rows as a vector-matrix multiplication, and compute the result.

        i. $5a_{2:} - 3a_{1:}$

        ii. $a_{1:} + a_{2:} + a_{3:}$

12. Consider two recipes calling for various amounts of three ingredients:

    - Recipe 1: 3 eggs, 25 g of sugar, and 2 cups of milk
    - Recipe 2: 5 eggs, 10 g of sugar, and 3 cups of milk

    Let the vector $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ be the quantity of each recipe you will make. Write the (egg,sugar,milk) requirements as a matrix-vector computation.

Vector Geometry

13. Find the value of v_2 that makes $v = \begin{bmatrix} .3 \\ v_2 \\ .5 \end{bmatrix}$ a unit vector.

14. Let $v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some angle θ.

    (a) Explain why v is a unit vector.

    (b) Explain why $w = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ is orthogonal to v.

15. Scale the vector $v = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ so that the result has length 10.
    i.e. find the "scalar" α such that $\|\alpha v\| = 10$.

16. Let $P(8, 2, 11)$ and $Q(13, 9, -5)$ be two points. Let $\vec{PQ}$ be the vector with initial point P and terminal point Q.

    (a) Find $\|\vec{PQ}\|$

    (b) Find the midpoint between P and Q by doing the vector calculation $\frac{1}{2}(P + Q)$.

    (c) Find the point 80% of the way (along a straight line) from P to Q.
    Hint: compute $P + .8\vec{PQ}$.

17. Let $A = \begin{bmatrix} 7 & 2 \\ c & 5 \end{bmatrix}$.

    (a) Find the value of c so that $a_1 \perp a_2$.

    (b) Find the value of c so that $a_{1:} \perp a_{2:}$.

18. Let $v = \begin{bmatrix} 4 \\ a \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 2a \\ a \\ 2 \end{bmatrix}$. If $v \perp w$, find the possible values of a.

19. Let $v = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 8 \end{bmatrix}$. If $w \perp v$ and $\|w\| = 9$, find $(v + w) \cdot (v + w)$.

20. Find a vector orthogonal to both columns of $A = \begin{bmatrix} 1 & 2 \\ 0 & 8 \\ 3 & 4 \end{bmatrix}$.

    Hint: let the desired vector look like $v = \begin{bmatrix} v_1 \\ 1 \\ v_3 \end{bmatrix}$.

21. Let $\vec{1}$ be the vector of all ones. Let $v = \begin{bmatrix} 6 \\ 2 \\ 9 \\ 3 \end{bmatrix} \in \mathbb{R}^4$.

    (a) Compute $\frac{\vec{1} \cdot v}{4}$.

(b) Compute $v - \left(\frac{\vec{1} \cdot v}{4}\right) \vec{1}$.

(c) Compute $v/(\vec{1} \cdot v)$.

Linear Systems

22. Sketch these lines and find the intersection.

$$x_1 + x_2 = 8 \qquad\qquad x_1 + 2x_2 = 14$$

Express the solution in the form $Ax = b$.

23. Find values of x_1 and x_2 such that:

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

Sketch the vector linear combination, and write that equation in the form $Ax = b$.

24. Find values of x_1 and x_2 such that:

$$x_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$$

Sketch the vector linear combination, and write that equation in the form $Ax = b$.

25. Find values of x_1 and x_2 such that:

$$x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

Sketch the vector linear combination, and write that equation in the form $Ax = b$.

26. A pet shop sells large birds for 60% more than they charge for for small birds. A lady bought five large and three small birds. If she had instead bought three large and five small, she would have saved $ 30. What is the price of each bird?

Write this problem and solution as a linear system $Ax = b$.

27. Below are linear systems written as $Ax = b$. For each one:

- write the Ax side as a linear combination
- write separate equations with $x_1, x_2, \cdots$
- solve the equations for x

(a) $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} x = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$

(b) $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

(d) $\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

(e) $\begin{bmatrix} 4 & -3 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$

(f) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

(g) $\begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

(h) $\begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

(i) $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -3 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

(j) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

(k) $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

28. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 36 \\ 19 \end{bmatrix}$$

29. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 11 \\ 43 \end{bmatrix}$$

30. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 3 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 33 \\ 52 \\ 24 \\ 109 \end{bmatrix}$$

31. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix}$$

Applications

32. Let $f(d,t)$ be a linear function giving the cost of d dolls and t trucks. If $f(0,2) = 9$ and $f(5,4) = 34$, find $f(2,3)$.

33. Jack is 6 years older than Jill. The sum of their ages is four times the age Jill was 8 years ago. How old are they?

34. You need three liters of a 30 percent alcohol solution. How can you mix 20 percent and 50 percent solutions to attain this?

35. Fifty-four cookies are to be fed to ten pets, some dogs and some cats. Each dog gets 7 cookies, and each cat gets 3. How many dogs and cats are there?

36. A movie theater sells tickets for \$ 8. The student discount is \$ 2. One night 525 tickets were sold for a total revenue of \$ 3580. How many regular and student tickets were sold?

37. Your smoothie stand offers three smoothies using these fruit ingredients:

- **abracadabra:**, \$ 3.50, 2 apples and 4 bananas
- **banana blast:** \$ 2.75, 6 bananas

- **citrus combo:** \$ 4.00, 1 apple, 2 bananas, 3 clementines

Each smoothie is 20 ounces. A customer's order is expressed as a vector of quantities q. Partial orders are allowed. e.g. $q_2 = 4.5$ means 90oz of banana blast.

(a) Write the vector p so that $p \cdot q$ is the total cost of the order.

(b) Write the matrix A such that Aq is how much of each fruit is required.

(c) A particular order requires 18 apples, 48 bananas, and 12 clementines. Find the order vector q.

(d) Find an order q that uses 10 apples, 25 bananas, and 15 clementines. How much does it cost?

(e) Find an order q that uses 10 apples, 15 bananas, and 20 clementines. Anything weird ?

38. Set up and solve a linear system that represents these chemical equations:

(a) $S_8 + F_2 \to SF_6$

(b) $C_5H_{12} + O_2 \to CO_2 + H_2O$

(c) $CO_2 + H_2O \to C_6H_{12}O_6 + O_2$ (photosynthesis)

39. Set up (but do not solve) a matrix equation $Ax = b$ for a linear system you could use to find:

    (a) the equation of a quadratic $f(x) = a + bx + cx^2$ that passes through $(0,3)$, $(4,2)$, and $(9,9)$.

    (b) the equation of a circle that passes through $(0,3)$, $(4,2)$, and $(9,9)$.

40. Explain the pattern for setting up a linear system to find:

    (a) the equation of a quadratic that passes through three given points

    (b) the equation of a circle that passes through three given points

41. Is it true? Explain if there are caveats.

    (a) There is a unique quadratic (parabola) $f(x) = a+bx+cx^2$ that fits through three distinct random points.

    (b) There is a unique circle that fits through three distinct random points.

Gauss Elimination

42. Solve the quadratic and circle systems above by equation substitution methods.

43. Solve the quadratic and circle systems above by Gauss Elimination. Show all elementary row operations to reduce the systems from $[A|b]$ to $[I|x]$. Use fractions.

44. For each augmented matrix, write and perform the next two row operations of the Gauss elimination algorithm for putting the system in RREF.

    (a) $\left[\begin{array}{cc|c} 1 & 7 & 3 \\ -4 & 2 & 5 \end{array}\right]$

    (b) $\left[\begin{array}{ccc|c} 1 & 7 & 3 & 3 \\ 0 & 2 & 3 & 9 \\ 0 & 8 & 5 & 4 \end{array}\right]$

    (c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right]$

    (d) $\left[\begin{array}{ccc|c} 5 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 3 & 6 \end{array}\right]$

    (e) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 6 \\ 0 & 3 & 1 & 7 \end{array}\right]$

    (f) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 6 & 7 \\ 0 & 0 & 1 & 7 & 9 \\ 0 & 3 & 2 & 3 & 4 \end{array}\right]$

45. Solve the triangular system by back-substitution without doing any additional row operations.

    (a) $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 2 & 8 \end{array}\right]$

    (b) $\left[\begin{array}{ccc|c} 3 & 7 & -4 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & \frac{1}{3} & 2 \end{array}\right]$

    (c) $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 5 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

    (d) $\left[\begin{array}{ccc|c} 0 & 0 & 2 & -8 \\ 0 & 1 & 2 & 4 \\ 6 & 2 & 3 & 0 \end{array}\right]$

    (e) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 6 \\ 1 & 2 & 3 & 0 & 7 \\ 1 & 2 & 3 & 4 & 8 \end{array}\right]$

    (f) $\left[\begin{array}{cccc|c} 1 & 0 & 6 & 0 & 5 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 3 & 0 & 7 \\ 0 & 0 & 0 & 4 & 8 \end{array}\right]$

46. This augmented matrix is equivalent to the given RREF.

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & b_1 \\ 1 & 8 & 3 & b_2 \\ 3 & -2 & 5 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Find the vector b.

47. Show all EROs to put this system in RREF.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 23 \\ 1 & 3 & 5 & 37 \\ 3 & 5 & 8 & 63 \end{array}\right]$$

48. Set up the augmented matrix representing a system you could solve to find the given curves. Then show all ERO's to reduce the system to RREF. Check your answer with Octave.

   (a) circle through $(1, 2)$, $(2, 0)$, and $(7, 1)$

   (b) parabola through $(1, 2)$, $(2, 0)$, and $(7, 1)$

   (c) cubic through $(0, 0)$, $(1, 2)$, $(2, 0)$, and $(7, 1)$

49. Use these three matrices to do the given computations, or say undefined.

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

(a) $A + 2B + I$
(b) $A + 2B^T + I$
(c) $3C$
(d) $C + D$
(e) $C + D^T$
(f) $\frac{1}{2}(A - A^T)$
(g) $E + 4E$
(h) $E + 4I$
(i) $E + E^T$

50. Find three different solutions to each of these linear systems (represented as augmented matrices)

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

(d) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$

51. Below is a "magic square". Solve the homogeneous system $Ax = \vec{0}$.

$$A = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

52. Below is a "magic square". Solve the homogeneous system $Ax = \vec{0}$ if $x_2 = 15$

$$A = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

53. If $D = ABC$, $D \in \mathbb{R}^{5\times 3}$, C is square, and A has 2 columns, then what is the size of B ?

54. Use these three matrices to do the given computations (by hand, but check with Octave), or say undefined.

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

(a) AB
(b) BA
(c) $AB - BA$
(d) $A^T A$
(e) AA^T
(f) A^2
(g) A^4
(h) $(A + B)^2$
(i) $A^2 + 2AB + B^2$
(j) AI
(k) IA
(l) $(A + 3I)^2$
(m) CD
(n) DC
(o) CE
(p) DE
(q) EC
(r) ED
(s) $ABCD$
(t) $ABDC$
(u) $A(B + DC)$
(v) ECA
(w) EAC
(x) CDE

55. Let A be a square matrix. Which of these must be symmetric?

(a) $A^T A$ (b) AA^T (c) $A + A^T$ (d) $A - A^T$

56. Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ be vectors. Compute the outer products xy^T and yx^T.

57. Express this matrix as an outer product xy^T. i.e. find x and y.

$$\begin{bmatrix} 2 & 6 & 4 \\ 3 & 9 & 6 \\ 7 & 21 & 14 \end{bmatrix}$$

58. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, and diagonal matrices $C = \mathrm{diag}(5, 2)$ and $D = \mathrm{diag}(2, 4, -1)$. Compute (or say undefined):

(a) AC (c) CA (e) CAD

(b) AD (d) DA (f) DAC

59. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

(a) Find a non-trivial solution to $Ax = \vec{0}$.
(b) Find a diagonal matrix D such that all the columns of AD are unit vectors.
(c) Find a diagonal matrix D such that all the rows of DA are unit vectors.
(d) Find a diagonal matrix D such that all the columns of AD have sum of 72.
(e) Find a diagonal matrix D such that all the rows of DA have sum of 72

60. Make up two diagonal matrices and multiply them together. Is the result diagonal ?

61. Make up two upper triangular matrices and multiply them together. Is the result upper triangular ?

62. Make up two lower triangular matrices and multiply them together. Is the result lower triangular ?

63. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Compute $A^{20} = A^{16}A^4$ by hand.

64. For each wealth transfer matrix (as described in class),

- Find $x^{(1)}$ and $x^{(2)}$ if the initial wealth is $x^{(0)} = 100(\vec{1})$.
- Solve the system $Ax = x$ to find the equilibrium distribution x. You may want to practice doing rref by hand, but check with Octave.

(a) $A = \begin{bmatrix} .3 & .9 \\ .7 & .1 \end{bmatrix}$

(b) $A = \begin{bmatrix} .1 & .2 & 0 \\ .4 & 0 & .6 \\ .5 & .8 & .4 \end{bmatrix}$

65. For each linear transformation:

   - Sketch the image of the e_i, e_j square (it will be a parallelogram).
   - Write a 2×2 matrix that accomplishes the linear transformation.
   - Check your answer at `https://integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php`

   (a) stretch horizontally by a factor of 3
   (b) reflect across the y-axis
   (c) rotate 90 degrees counter-clockwise
   (d) rotate $\pi/5$ radians
   (e) shear horizontally with a factor of 2
   (f) shear vertically with a factor of $\frac{1}{2}$

66. For each matrix representing a linear transformation:

   - Sketch the image of the e_i, e_j square (it will be a parallelogram).
   - Describe the geometric effect in words.
   - Check your answer at `https://integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php`

   (a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$
   (b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
   (d) $\begin{bmatrix} .866 & -.5 \\ .5 & .866 \end{bmatrix}$
   (e) $\begin{bmatrix} .5 & .866 \\ -.866 & .5 \end{bmatrix}$
   (f) $\begin{bmatrix} -.5 & -.866 \\ .866 & -.5 \end{bmatrix}$

67. Let $R = \begin{bmatrix} .75 & * \\ * & * \end{bmatrix}$ be a rotation matrix. What are the two possibilities for R ?

68. Write out and perform matrix multiplication to find the matrix representing these combinations of linear transformations. Indicate which ones work out to be the same thing regardless of which order they are applied (i.e. the matrices commute)

   (a) stretch by factor of 5 horizontally, then stretch by a factor of 8 vertically
   (b) rotate 30 degrees counter-clockwise, then rotate 90 degrees counter-clockwise
   (c) rotate 30 degrees counter-clockwise, then reflect across the x-axis
   (d) rotate 30 degrees counter-clockwise, then stretch by a factor of 3 horizontally
   (e) rotate 30 degrees counter-clockwise, then shear horizontally with a factor of 2
   (f) shear horizontally with a factor of 2, then shear vertically with a factor of 3
   (g) shear horizontally with a factor of 2, then stretch by a factor of 3 vertically
   (h) shear horizontally with a factor of 2, then reflect across the x-axis
   (i) reflect across the x-axis, then reflect across the y-axis

69. Multiply matrices to accomplish the given sequence (you may use Octave)

   (a) rotate 30 degrees counter-clockwise, then reflect across the x-axis, then shear vertically with a factor of 3
   (b) stretch by a factor of 5 horizontally, then rotate 30 degrees counter-clockwise, then shear horizontally with a factor of 2, then reflect across the y-axis.

70. What is the net effect of:

(a) stretch by a factor of 2 horizontally, then stretch by a factor of $\frac{1}{2}$ horizontally
(b) rotate 30 degrees counter-clockwise, then rotate 30 degrees clockwise
(c) reflect across the x-axis, then reflect across the x-axis
(d) shear horizontally with a factor of 2, then shear horizontally with a factor of -2

71. Let A be a 5×5 matrix with $a_{ij} = 5(i-1)+j$. Write/evaluate the following:

(a) A

(b) Ae_3

(c) $A^T e_3$

(d) $e_2^T A$

(e) $e_2^T A^T$

(f) $e_2^T A e_3$

(g) $e_2^T A^T e_3$

(h) $e_2^T A \vec{1}$

(i) $\vec{1}^T A e_2$

72. Find the following:

(a) rotation $R = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$, find R^{-1}

(b) scaling $D = \begin{bmatrix} .25 & 0 \\ 0 & 1.25 \end{bmatrix}$, find D^{-1}

(c) shear $S = \begin{bmatrix} 1 & 0 \\ .75 & 1 \end{bmatrix}$, find S^{-1}

(d) the composition $T = DR$, find T and T^{-1}

(e) the composition $T = RD$, find T and T^{-1}

(f) the composition $T = RS$, find T and T^{-1}

(g) the composition $T = RDS$, find T and T^{-1}

73. Compute $\det(A)$, and if possible, find A^{-1}.

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 9 & -5 \\ 7 & 4 \end{bmatrix}$

(d) $A = \begin{bmatrix} 9 & 3 \\ 6 & 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}$

(f) $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

74. The matrix $A = \begin{bmatrix} x & 5 \\ x-2 & 8 \end{bmatrix}$ is not invertible. Find the value of $x \in \mathbb{R}$.

75. Let $A^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix}$.

(a) If $Ax = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$, then find $x \in \mathbb{R}^2$.

(b) Is it true that $(A^{-1})^{-1} = A$?

(c) Compute $A^2 e_2$.

76. If $A^2 = I$, then A is its own inverse. Find three examples of 2×2 matrices with this property.

77. Let $A = \begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}$. Verify that $(A^T)^{-1} = (A^{-1})^T$.

78. Let $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The matrix $M \in \mathbb{R}^{2\times 2}$ is a mystery matrix. Solve for M if:

(a) $AM = I$

(b) $MA = I$

(c) $AM = B$

(d) $MA = B$

(e) $AM = A$

(f) $BM = A$

(g) $AMB = I$

(h) $AMB = A$

(i) $AMA = A$

(j) $AMA = B$

79. Let $A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$. Compute A^k for $k = -3 - 2, -1, 0, 1, 2, 4, 8$

80. Let A be a square matrix. If $M^2 = A$, we may write $M = \sqrt{A}$. Find square roots of these matrices:

   (a) rotation $R = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$

   (b) scaling $D = \begin{bmatrix} .25 & 0 \\ 0 & 1.25 \end{bmatrix}$

   (c) shear $S = \begin{bmatrix} 1 & 0 \\ .75 & 1 \end{bmatrix}$

81. Wealth transfers between two entities according to this model:

$$x^{(k+1)} = Ax^{(k)}$$

If $A = \begin{bmatrix} .2 & .7 \\ .8 & .3 \end{bmatrix}$, and $x^{(3)} = \begin{bmatrix} 52 \\ 48 \end{bmatrix}$, find $x^{(0)}$.

82. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Compute this ratio for large n. Can you guess what number it approaches ?

$$\frac{e_1 A^n e_1}{e_2^T A^n e_1}$$

83. Let $A = \begin{bmatrix} 8 & 32 \\ 15 & 50 \end{bmatrix}$.

   (a) Find a rotation R so that $RA = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

   (b) Find a diagonal D so that $RAD = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

   (c) Find a shear S so that $SRAD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (d) Factor A as a product of three matrices: $A = R^{-1}S^{-1}D^{-1}$.

   (e) Verify that $A^{-1} = DSR$

84. Let $A = \begin{bmatrix} 8 & 32 \\ 15 & 50 \end{bmatrix}$.

   (a) Find a shear S_1 so that $S_1 A = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

   (b) Find a shear S_2 so that $S_2 S_1 A = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$

(c) Find a diagonal D so that $DS_2S_1A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(d) Verify that $A^{-1} = DS_2S_1$

(e) Factor A as a product of three matrices.

85. Experiment with 2×2 matrices to find a counter-example to each of these statements.

    (a) If A and B are invertible, then $(A+B)^{-1} = A^{-1} + B^{-1}$

    (b) If $AB = 0$, then either $A = 0$ or $B = 0$.

    (c) If $AB = AC$, then $B = C$.

    (d) If $Ax = 0$ then $A^T x = 0$.

86. For each matrix, if possible:

    - Factor as $A = QR$, where $Q^TQ = I$ and R is upper triangular.
    - Factor as $A = LU$, where L is lower triangular, and U is upper triangular.

    (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

    (b) $A = \begin{bmatrix} -3 & 1 \\ 5 & 7 \end{bmatrix}$

    (c) $A = \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}$

    (d) $A = \begin{bmatrix} 0 & 8 \\ 2 & 4 \end{bmatrix}$

87. Let $A = \begin{bmatrix} 3 & -12 \\ 2 & 7 \end{bmatrix}$. This sequence of row operations produces the equivalent RREF:

    - divide row 1 by 3
    - subtract 2 times row 1 from row 2
    - scale row 2 by 1/15
    - add 4 times row 2 to row 1

    (a) Write each row operation as a matrix multiplication, so that $M_4M_3M_2M_1A = I$.

    (b) Write A^{-1} as the product of four matrices.

    (c) Write A as the product of four matrices.

88. Find the angle between the vectors $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

89. Find two values of y so that the angle between $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ y \end{bmatrix}$ is 30°.

90. Find the angle between the vectors $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$.

91. Find the angle between the lines $y = 2x$ and $y = 5 - 3x$.

92. For each vector v,

- Find the matrix P for the orthogonal projection onto v.
- Find the reflection matrix R for the orthogonal reflection across v.

(a) $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(b) $v = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

(c) $v = \begin{bmatrix} 45 \\ -28 \end{bmatrix}$

(d) $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(e) $v = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

(f) $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

93. Find the matrix P that orthogonally projects vectors onto the line $y = \frac{7}{3}x$.

94. Find the matrix R that orthogonally reflects vectors across the line $y = \frac{7}{3}x$.

95. Find the matrix that represents the composition:

(a) first rotate 45°, then project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

(b) first project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$, then rotate 45°

(c) first scale by $\text{diag}(3, 2)$, then project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

(d) first project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$, then scale by $\text{diag}(3, 2)$

96. Given an orthogonal projection $P = \frac{vv^T}{v^Tv}$ onto v, let $F(\alpha) = (1 - \alpha)P + \alpha I$ be the corresponding flattening operator.

(a) Find the inverse of $F(0.5)$.

(b) Find the inverse of $F(0.2)$.

(c) Find the inverse of $F(-1)$.

97. Let $x \in \mathbb{R}^n$, and $\vec{1} \in \mathbb{R}^n$ be the vector of all ones.

(a) Write the projection matrix P onto $\vec{1}$.

(b) If $x = \begin{bmatrix} 5 \\ 9 \\ 6 \\ 4 \end{bmatrix}$, find Px. Can you describe what Px is doing ?

(c) Show algebraically that $1^T Px = 0$ for all vectors x.

98. Find the determinant of each matrix by hand (you can either row reduce to an upper triangular, or use cofactor expansion)

(a) $\begin{bmatrix} 5 \end{bmatrix}$

(b) $\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} 5 & 3 & 1 & 3 \\ 1 & 2 & 4 & 8 \\ 4 & 3 & -2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ 8 & 2 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} 8 & 2 & 3 \\ 1 & 5 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 2 & 3 & 0 & 2 \\ 2 & 6 & -1 & 5 \\ 4 & 12 & -3 & 7 \\ 2 & 3 & -1 & -1 \end{bmatrix}$

(h) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

(j) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$

(k) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

(l) $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

99. Is the statement generally true or false ? (assume A and B are $n \times n$ matrices).

(a) $\det(AB) = \det(BA)$

(b) If A is singular, then $\det(A) = 0$.

(c) $\det(A + B) = \det(A) + \det(B)$

(d) $\det(-A) = -\det(A)$

(e) $\det(A^{-1})\det(A) = 1$ if A is invertible

(f) $\det(A^T A) = \det(A)^2$

(g) $\det(3A) = 3^n \det(A)$

(h) $\det(A + 2I) = \det(A) + 2$

(i) If you swap two rows of A, the determinant doesn't change.

(j) If rows of A satisfy $a_{1:} + a_{2:} = a_{3:}$, then $\det(A) = 0$.

(k) If columns of A satisfy $a_1 + a_2 = a_3$, then $\det(A) = 0$.

(l) If $A^T = -A$ and n is odd, then $\det(A) = 0$.

100. Find all possible values of x.

(a) $\det \begin{bmatrix} 3 & 4 \\ 2 & x \end{bmatrix} = 12$

(b) $\det \begin{bmatrix} x & 4 \\ 2 & x \end{bmatrix} = 12$

(c) $\det \begin{bmatrix} x & 4 \\ x & x \end{bmatrix} = 12$

(d) $\det \begin{bmatrix} x & 4 \\ x & x \end{bmatrix} = x$

(e) $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 \\ 1 & x \end{bmatrix} = \det \begin{bmatrix} 2 & 2 \\ 4 & 4+x \end{bmatrix}$

(f) $\det \left(\begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix} + xI \right) = 0$

(g) $\det \left(\begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix} + xI \right) = 12$

(h) $\det \begin{bmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{bmatrix} = 0$

(i) $\det \begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix} = 0$

101. Find the adjoint of these matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

102. If A has all integer entries, and $\det(A) = \pm 1$, then A^{-1} has all integer entries. To produce such a matrix, start with a triangular matrix with 1's on the Gordon line, and do random row operations (using integers). Let's illustrate:

(a) Start with $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Do some row operations, and write the resulting matrix A.

- $R_2 \mathrel{+}= 3R_1$
- $R_3 \mathrel{-}= 2R_2$
- swap rows 1 and 3
- $R_3 \mathrel{-}= R_1$

(c) Check that $\det(A)$ is either 1 or -1.

(d) Find A^{-1} using the adjoint formula.

103. Create a 4×4 matrix A of integers, with no zero entries, having $\det(A) = 20$.
Hint: start with a triangular matrix with the desired determinant, then do row operations to fill in the zeros.

104. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$.

(a) Find $u \times v$.

(b) Find $v \times u$.

(c) Find $(2u) \times v$.

(d) Find $(u + v) \times v$.

(e) Find $u \times u$.

105. Find the area of the triangles with given corners.

(a) $P(0,0)$, $Q(-2,9)$, $R(5,3)$

(b) $P(8,2)$, $Q(15,5)$, $R(7,10)$

(c) $P(0,0,0)$, $Q(-2,9,4)$, $R(5,3,12)$

(d) $P(8,2,3)$, $Q(15,5,11)$, $R(7,10,6)$

106. The volume of a triangular pyramid (also known as a tetrahedron) is $\frac{1}{6}|\det(A)|$ where the columns of A are the edges emanating from one corner (which we can place at the origin). Find the volume of such a pyramid if the edges are $u = \begin{bmatrix} 10 \\ 6 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 12 \\ 3 \end{bmatrix}$, and $w = \begin{bmatrix} 5 \\ 5 \\ 20 \end{bmatrix}$.

107. Sketch the pentagon with corners at $(0,0)$, $(-1,3)$, $(4,6)$, $(5,2)$, and $(4,0)$. Divide it into 3 triangles, and find the area of the pentagon.

108. Let $u = \begin{bmatrix} x \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.
Find the two values of x so that the parallogram determined by u and v has area 100.

109. Find the volume of the parallelpiped determined by the columns of $A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$.

110. Let $A = \begin{bmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{bmatrix}$ determine a parallelpiped.

(a) Find the volume if $x = 0$.

(b) Find the volume if $x = 1$.

(c) Find the volume if $x = 2$.

(d) Find the volume if $x = 3$.

(e) Determine the value of x such that the volume is 100.

111. Let $u = \begin{bmatrix} 6 \\ 17 \\ 10 \end{bmatrix}$ and $v = \begin{bmatrix} 21 \\ 5 \\ 32 \end{bmatrix}$.
Find a vector w that is orthogonal to both u and v, scaled so that $\|w\| = 50$.

112. Find two values of λ such that $\det \begin{bmatrix} 4-\lambda & 6 \\ 6 & 4-\lambda \end{bmatrix} = 0$

113. Let $v \in \mathbb{R}^n$ (with $n > 1$) be any unit vector. Then $P = vv^T$ is a matrix that projects onto v. Using an example, argue that $\det(P) = 0$.

114. Suppose $Q^T Q = I$. Argue that $\det(Q) = \pm 1$.

115. By hand, for each matrix:

- Compute $\det(A - \lambda I)$. This will give you a **characteristic polynomial** in λ.
- Find the roots (set it equal to zero), which are the eigenvalues.
- For each eigenvalue, find an eigenvector.

(a) $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$

(d) $A = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$

(e) $A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$

(f) $A = \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$

(g) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

(h) $A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$

(i) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ (try to find the eigenvectors "by inspection")

(j) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

(k) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

(l) $A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 8 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$

116. Let $A = \begin{bmatrix} 4 & b \\ 3 & 6 \end{bmatrix}$. Find the value(s) of b such that the eigenvalues are $\lambda = 2, 8$.

117. Let $A = \begin{bmatrix} 4 & b \\ b & 6 \end{bmatrix}$. Find the value(s) of b such that the eigenvalues are $\lambda = 2, 8$.

118. Suppose A is 3×3. If $\det(A) = 80$, the trace of A is 1, and $A + 2I$ is singular, then find the eigenvalues of A.

119. Suppose the eigenvalues of A are $\lambda = 2, 8$.

(a) Find the eigenvalues of $.5A$.
(b) Find the eigenvalues of $A - 3I$.
(c) Find the eigenvalues of $.5A - 3I$.

120. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 7 \end{bmatrix}$. Find the characteristic polynomial $\det(A - \lambda I)$, and show that it has no real roots.

121. Let $A = \begin{bmatrix} -237 & 66 & -243 & 211 \\ 540 & -129 & 456 & -472 \\ 12 & -2 & 10 & -11 \\ -404 & 108 & -390 & 357 \end{bmatrix}$.
For each vector x, determine if x is an eigenvector, and if so, find the eigenvalue (by hand).

(a) $x = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 3 \end{bmatrix}$ (b) $x = \begin{bmatrix} 7 \\ -4 \\ 1 \\ 10 \end{bmatrix}$ (c) $x = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 5 \end{bmatrix}$ (d) $x = \begin{bmatrix} 4 \\ 2 \\ 1 \\ 5 \end{bmatrix}$

Now, what is the fourth eigenvalue ? (hint: use the trace)

122. Let $A = \begin{bmatrix} 4 & * \\ * & 7 \end{bmatrix}$, where $*$ are unknown entries. If $\det(A) = 3$, find the two eigenvalues of A.

123. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 9 & 6 \\ 0 & x & 2 \end{bmatrix}$. Where x is an unknown entry. Let λ_1 denote the largest eigenvalue of A.

(a) If $x = 0$, find $\det(A)$ and λ_1.
(b) If $x = 1$, find $\det(A)$ and λ_1.
(c) If $x = 2$, find $\det(A)$ and λ_1.
(d) If $x = 3$, find $\det(A)$ and λ_1.
(e) Find $\det(A)$ as a function of x.
(f) Find λ_1 as a function of x (that works for $x \geq 0$).
(g) If $\lambda_1 = 50$, find $\det(A)$.

124. Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ by hand, and check with Octave.

125. Let $P = \frac{vv^T}{v^Tv}$ be a projection matrix. Let $w \perp v$. Then from the notes we know that the eigenpairs of P are $\{(1, v), (0, w)\}$.

(a) Let $R = 2P - I$ be the reflection matrix (across v). Find the eigenpairs of R.
(b) Let $F = .25P + .75I$. Find the eigenpairs of F.

126. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 6 & 0 & 0 \end{bmatrix}$. This is not referred to as a "triangular" matrix because the triangle cuts the wrong way. Is it still true that the eigenvalues are the diagonal entries ?

127. Let $A = \begin{bmatrix} 1 & 11 & -26 \\ 11 & 64 & 37 \\ -26 & 37 & 49 \end{bmatrix}$. The eigenvalues of A are $\lambda = -19, 38, 95$.

(a) By doing rref (use Octave) of $\lambda + 19I$, find an eigenvector that goes with $\lambda = -19$.

(b) Find an eigenvector that goes with $\lambda = 38$.

(c) Find an eigenvector that goes with $\lambda = 95$.

(d) Notice that A is symmetric. Verify that the three eigenvectors you found are all orthogonal to each other.

128. The 2×2 matrix A that has these eigenpairs:

$$\left(3, \begin{bmatrix} 5 \\ 2 \end{bmatrix}\right), \qquad \left(7, \begin{bmatrix} 3 \\ 8 \end{bmatrix}\right)$$

(a) Put the eigenvectors in V, and the eigenvalues in a diagonal matrix Λ, and find $A = V\Lambda V^{-1}$.

(b) Find the eigenvectors of A^T.

129. Consider matrices of the form $V\Lambda V^{-1}$ where

$$V = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

(a) Find V^{-1}.

(b) Let A be the result if $\lambda_1 = 9$ and $\lambda_2 = 4$.

i. Compute $A = V\Lambda V^{-1}$.

ii. Verify that $A^2 = V\Lambda^2 V^{-1}$.

iii. Verify that $A^{-1} = V\Lambda^{-1} V^{-1}$.

iv. Compute $S = V \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} V^{-1}$, and verify that $S^2 = A$.

(c) Let B be the result if $\lambda_1 = 0.6$ and $\lambda_2 = -0.8$.

i. Compute $B = V\Lambda V^{-1}$.

ii. Compute $B^{99} = V\Lambda^{99} V^{-1}$. Why is it almost zero ?

(d) Let C be the result if $\lambda_1 = 0.2$ and $\lambda_2 = -1.1$.

i. Compute $C = V\Lambda V^{-1}$.

ii. Compute $C^{99} = V\Lambda^{99} V^{-1}$. Why is it blowing up ?

130. Here is a transition matrix for a **Markov chain**, so that $x^{(k+1)} = Tx^{(k)}$.

$$T = \begin{bmatrix} * & .12 & 0 & .31 & .25 \\ .19 & * & .27 & .22 & .10 \\ .32 & .24 & * & .13 & .27 \\ .14 & .12 & .24 & * & .08 \\ .15 & .37 & .21 & .34 & * \end{bmatrix}$$

(a) Enter the matrix into Octave, filling in the diagonal so that each column sums to 1.

(b) Use the command `[V,L] = eig(T)` to find the eigenvalues and vectors.

(c) For $\lambda = 1$, find an eigenvector x, so that $Tx = x$ and $1^T x = 500$ (i.e. the entries of x sum to 500).

(d) Compute T^5. What do you notice about it ?

(e) Suppose the first entity started with all the money, i.e. $x^{(0)} = 500e_1$. Find $x^{(5)} = T^5 x^{(0)}$.

(f) Suppose the last entity started with all the money, i.e. $x^{(0)} = 500e_5$. Find $x^{(5)} = T^5 x^{(0)}$.