

Homework - Spring 2020

Vector, Matrix Notation

1. Let $x = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

- (a) State the dimension of these vectors.

Answer: 3

- (b) Find x_2 .

Answer: $x_2 = 9$

- (c) Write the transpose x^T as a row vector.

Answer: $[5 \ 9 \ 2]$

- (d) Write $(x^T)^T$.

Answer: the original column vector

- (e) Compute the dot product $x \cdot y$.

Answer: $(5)(7) + (9)(2) + (2)(3) = 59$

- (f) Compute $y \cdot x$.

Answer: 59

- (g) Change the value of y_3 to force $x \cdot y = 0$.

Answer: $35 + 18 + 2y_3 = 0$, so $y_3 = -53/2$

- (h) Consider a vector of ones: $\vec{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Compute $\vec{1} \cdot x$. What is this accomplishing?

Answer: $5 + 9 + 2 = 16$, sums the vector x

- (i) Compute the linear combination $5x - 3y$.

Answer: $\begin{bmatrix} 4 \\ 39 \\ 1 \end{bmatrix}$

- (j) Compute $\vec{1} \cdot (5x - 3y)$. Does it equal $5(\vec{1} \cdot x) - 3(\vec{1} \cdot y)$?

Answer: yes, $44 = 5(16) - 3(12)$

2. Let $A = \begin{bmatrix} 2 & 5 & 1 & 8 & 7 \\ 4 & 9 & 7 & 0 & 2 \\ 0 & 2 & 1 & -6 & 3 \end{bmatrix}$.

- (a) State the $m \times n$ dimensions of this matrix.

Answer: 3×5

- (b) Find a_{23} .

Answer: 7

- (c) Write the indices of the entries with $a_{ij} = 0$.

Answer: a_{31} and a_{24}

- (d) Write $a_{:,2}$, i.e. the 2nd column.

Answer: $\begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$

- (e) Write $a_{3,:}$, i.e. the 3rd row.

Answer: $[0 \ 2 \ 1 \ -6 \ 3]$

(f) Compute $a_{:2} \cdot a_{:5}$

Answer: 59

(g) Compute $a_{2:} \cdot a_{3:}$

Answer: 31

(h) Write the transpose matrix A^T .

Answer: $A^T = \begin{bmatrix} 2 & 4 & 0 \\ 5 & 9 & 2 \\ 1 & 7 & 1 \\ 8 & 0 & -6 \\ 7 & 2 & 3 \end{bmatrix}$

(i) Is it always true that $(A^T)^T = A$?

Answer: yes

Linear Functions

3. A worker makes \$ 9 per hour. Let $f(t)$ be the wages earned in t hours. Is $f(t)$ linear ? Compute $f(30)$.

Answer: $f(t) = 9t$, so it is linear, and $f(30) = 270$

4. Suppose you deposit \$ 1000 in a bank account pays 4% annual interest. You leave the money alone and let it compound. Let $f(t)$ be the total amount of interest earned after t years. Compute $f(1)$ and $f(2)$. Is $f(t)$ linear ?

Answer: no, $f(1) = 40$ and $f(2) = 81.6$, so f is non-linear

5. Let $f(x)$ be your profit if you charge x dollars for your fund raiser's candy bars. Is $f(x)$ linear ?

Answer: no, $f(x)$ would probably have a parabolic shape, rising to a max and then falling if you charge too much

6. A recipe calls for $\frac{2}{3}$ cup of sugar to make a dozen cookies. Let $f(x)$ be the amount of sugar needed to make x cookies. Is $f(x)$ linear ?

Answer: $f(x) = \frac{1}{18}x$ is linear

7. A batch of cookies requires $\frac{2}{3}$ cup of sugar, and a batch of brownies requires 2 cups of sugar. If you make b batches of brownies, and c batches of cookies, write your sugar requirement as a linear function $f(b, c)$.

Answer: $2b + \frac{2}{3}c$

8. An apple costs 75¢, and a banana is 40¢. Let the bill be $f(a, b)$, where a and b are the number of each fruit purchased.

(a) Xerxes buys 3 apples and 5 bananas; how much is his bill $f(3, 5)$?

Answer: $\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 75 \\ 40 \end{bmatrix} = (3)(75) + (5)(40) = 425$

(b) Yolanda buys 4 apples and 8 bananas; how much is her bill $f(4, 8)$?

Answer: $\begin{bmatrix} 4 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} 75 \\ 40 \end{bmatrix} = (4)(75) + (8)(40) = 620$

(c) Xerxes offers to pay for all the fruit; what is the total bill $f(7, 13)$?

Answer: $\begin{bmatrix} 7 \\ 13 \end{bmatrix} \cdot \begin{bmatrix} 75 \\ 40 \end{bmatrix} = (7)(75) + (13)(40) = 1045$

(d) Is it true that $f(a_1, b_1) + f(a_2, b_2) = f(a_1 + a_2, b_1 + b_2)$?

Answer: yes

(e) Let $k \in \mathbb{N}$ be a constant. Is it true that $f(ka, kb) = kf(a, b)$?

Answer: yes, e.g. double the purchase double the bill

9. You own a portfolio of 100 shares of AAPL, 60 shares of GOOG, and 250 shares of MSFT.

(a) Write the value of your portfolio a linear function of the current prices per share: $f(a, g, m)$.

Answer: $f(a, g, m) = 100a + 60g + 250m$

(b) If all three stocks rise by 2% on bullish news for the tech industry, does the value of your portfolio go up by 2% ? i.e. is it true that $f(1.02a, 1.02g, 1.02m) = 1.02f(a, g, m)$?

Answer: yes

Linear Combinations

10. Let $x = \begin{bmatrix} 5 \\ 9 \\ 2 \end{bmatrix}$ and $y = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

(a) Find $x \cdot y$, $x \cdot x$, and $y \cdot y$.

Answer: 59, 110, 62

(b) Find $\|x\|$ and $\|y\|$.

Answer: $\sqrt{110}, \sqrt{62}$

(c) Find $\|-3x\|$. What can you say about $\|cx\|$ for $c \in \mathbb{R}$?

Answer: $\sqrt{(-15)^2 + (-27)^2 + (-6)^2} = \sqrt{990} = 3\|x\|$, and generally $\|cx\| = |c|\|x\|$

(d) Find $\|x + y\|$. Is it less than, equal to, or greater than $\|x\| + \|y\|$?

Answer: $\sqrt{12^2 + 11^2 + 5^2} = \sqrt{290}$ which is less than $\sqrt{110} + \sqrt{62}$

(e) Find $x \cdot (3x - y)$. Does the distributive property work?

Answer: $x \cdot \begin{bmatrix} 8 \\ 25 \\ 3 \end{bmatrix} = 271$, and $3(x \cdot x) - (x \cdot y) = 330 - 59 = 271$, so yes dot product distributes

(f) Find $(3x - y) \cdot (3x - y)$. Express it in terms of $\|x\|$, $\|y\|$, and $x \cdot y$

Answer: $9x \cdot x - 3x \cdot y - 3y \cdot x + y \cdot y = 9\|x\|^2 + \|y\|^2 - 6(x \cdot y) = 698$

(g) Find $\|3x - y\|$.

Answer: $\sqrt{698}$

(h) Write $3x - y$ as a matrix-vector multiplication.

Answer: $\begin{bmatrix} 5 & 7 \\ 9 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$

11. Let $A = \begin{bmatrix} 5 & 4 & 5 & 7 \\ 9 & 7 & 8 & 2 \\ 2 & 1 & -1 & 3 \end{bmatrix}$.

(a) Write these linear combinations of A's columns as a matrix-vector multiplication, and compute the result.

i. $2a_2 + 7a_4$

Answer: $A \begin{bmatrix} 0 \\ 2 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 57 \\ 28 \\ 23 \end{bmatrix}$

ii. $5a_2 - 3a_1 - a_3$

Answer: $A \begin{bmatrix} -3 \\ 5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(b) Write these linear combinations of A's rows as a vector-matrix multiplication, and compute the result.

i. $5a_2 - 3a_1$:

Answer: $[-3, 5, 0]A = [30, 23, 25, -11]$

ii. $a_1 + a_2 + a_3$:

Answer: $[1, 1, 1]A = [16, 12, 12, 12]$

12. Consider two recipes calling for various amounts of three ingredients:

- Recipe 1: 3 eggs, 25 g of sugar, and 2 cups of milk
- Recipe 2: 5 eggs, 10 g of sugar, and 3 cups of milk

Let the vector $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ be the quantity of each recipe you will make. Write the (egg,sugar,milk) requirements as a matrix-vector computation.

Answer: $\begin{bmatrix} 3 & 5 \\ 25 & 10 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$

Vector Geometry

13. Find the value of v_2 that makes $v = \begin{bmatrix} .3 \\ v_2 \\ .5 \end{bmatrix}$ a unit vector.

Answer: $\sqrt{.66}$

14. Let $v = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ for some angle θ .

(a) Explain why v is a unit vector.

Answer: $\|v\| = \cos^2 \theta + \sin^2 \theta = 1$

(b) Explain why $w = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$ is orthogonal to v .

Answer: dot product is zero

15. Scale the vector $v = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$ so that the result has length 10.

i.e. find the “scalar” α such that $\|\alpha v\| = 10$.

Answer: $\alpha = \frac{10}{\sqrt{74}}$

16. Let $P(8, 2, 11)$ and $Q(13, 9, -5)$ be two points. Let \vec{PQ} be the vector with initial point P and terminal point Q .

(a) Find $\|\vec{PQ}\|$

Answer: $\sqrt{5^2 + 7^2 + 16^2} = \sqrt{330}$

(b) Find the midpoint between P and Q by doing the vector calculation $\frac{1}{2}(P + Q)$.

Answer: $(10.5, 5.5, 3)$

(c) Find the point 80% of the way (along a straight line) from P to Q .

Hint: compute $P + .8\vec{PQ}$.

Answer: $(12, 7.6, -1.8)$

17. Let $A = \begin{bmatrix} 7 & 2 \\ c & 5 \end{bmatrix}$.

(a) Find the value of c so that $a_1 \perp a_2$.

Answer: $14 + 5c = 0$, so $c = -14/5$

(b) Find the value of c so that $a_1 \cdot a_2 = 0$.

Answer: $7c + 10 = 0$, so $c = -10/7$

18. Let $v = \begin{bmatrix} 4 \\ a \\ 3 \end{bmatrix}$ and $w = \begin{bmatrix} 2a \\ a \\ 2 \end{bmatrix}$. If $v \perp w$, find the possible values of a .

Answer: $a^2 + 8a + 6 = 0$ by QF $a = -4 \pm \sqrt{10}$

19. Let $v = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 8 \end{bmatrix}$. If $w \perp v$ and $\|w\| = 9$, find $(v + w) \cdot (v + w)$.

Answer: $\|v\| = \sqrt{102}$. By the PT, $\|v + w\|^2 = \|v\|^2 + \|w\|^2 = 102 + 81 = 183$

20. Find a vector orthogonal to both columns of $A = \begin{bmatrix} 1 & 2 \\ 0 & 8 \\ 3 & 4 \end{bmatrix}$.

Hint: let the desired vector look like $v = \begin{bmatrix} v_1 \\ 1 \\ v_3 \end{bmatrix}$.

Answer: we want $v_1 + 3v_3 = 0$ and $2v_1 + 8 + 4v_3 = 0$, substitute to get $v_3 = 4$, so $v = \begin{bmatrix} -12 \\ 1 \\ 4 \end{bmatrix}$

21. Let $\vec{1}$ be the vector of all ones. Let $v = \begin{bmatrix} 6 \\ 2 \\ 9 \\ 3 \end{bmatrix} \in \mathbb{R}^4$.

- (a) Compute $\frac{\vec{1} \cdot v}{4}$.

Answer: 5, note this is the arithmetic mean of v 's entries

- (b) Compute $v - \left(\frac{\vec{1} \cdot v}{4}\right) \vec{1}$.

Answer: $[1; -3; 4; -2]$

- (c) Compute $v / (\vec{1} \cdot v)$.

Answer: $[\.3; .1; .45; .15]$

Linear Systems

22. Sketch these lines and find the intersection.

$$x_1 + x_2 = 8$$

$$x_1 + 2x_2 = 14$$

Express the solution in the form $Ax = b$.

Answer: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

23. Find values of x_1 and x_2 such that:

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$$

Sketch the vector linear combination, and write that equation in the form $Ax = b$.

Answer: $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix}$

24. Find values of x_1 and x_2 such that:

$$x_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$$

Sketch the vector linear combination, and write that equation in the form $Ax = b$.

Answer: $\begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 20 \end{bmatrix}$

25. Find values of x_1 and x_2 such that:

$$x_1 \begin{bmatrix} 3 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$$

Sketch the vector linear combination, and write that equation in the form $Ax = b$.

Answer: $\begin{bmatrix} 3 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix}$

26. A pet shop sells large birds for 60% more than they charge for for small birds. A lady bought five large and three small birds. If she had instead bought three large and five small, she would have saved \$ 30. What is the price of each bird?

Write this problem and solution as a linear system $Ax = b$.

Answer: $5L + 3S = 20 + 3L + 5S$ and $L = 1.6S$

$$\begin{bmatrix} 2 & -2 \\ 1 & -1.6 \end{bmatrix} \begin{bmatrix} 40 \\ 25 \end{bmatrix} = \begin{bmatrix} 30 \\ 0 \end{bmatrix}$$

27. Below are linear systems written as $Ax = b$. For each one:

- write the Ax side as a linear combination
- write separate equations with x_1, x_2, \dots
- solve the equations for x

(a) $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} x = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$
Answer: $[3; 2/3]$

(b) $\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$
Answer: $[3/2; 4/3]$

(c) $\begin{bmatrix} 4 & 2 \\ 0 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$
Answer: $[-1/2; 4]$

(d) $\begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$
Answer: $[3/2; 3]$

(e) $\begin{bmatrix} 4 & -3 \\ 2 & 3 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$
Answer: $[3; 2]$

(f) $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$
Answer: $[3/2; 4; 2]$

(g) $\begin{bmatrix} 4 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

Answer: $[-1/2; 4; 2]$

(h) $\begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

Answer: $[-2; 6; 2]$

(i) $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & -3 & 5 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

Answer: $[3/2; 3; 7/2]$

(j) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

Answer: $[6; 12; 10]$

(k) $\begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} 6 \\ 12 \\ 10 \end{bmatrix}$

Answer: $[18; -8; 10]$

28. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 27 \\ 36 \\ 19 \end{bmatrix}$$

Answer: 9

29. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ 11 \\ 43 \end{bmatrix}$$

Answer: 5

30. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 3 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \\ 3 \end{bmatrix} = \begin{bmatrix} 33 \\ 52 \\ 24 \\ 109 \end{bmatrix}$$

Answer: 7

31. Find the value of x_3 that solves:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ x_3 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 24 \\ 8 \end{bmatrix}$$

Answer: -4

Applications

32. Let $f(d, t)$ be a linear function giving the cost of d dolls and t trucks. If $f(0, 2) = 9$ and $f(5, 4) = 34$, find $f(2, 3)$.

Answer: first see that each truck costs \$4.50. Then find that a doll is \$3.20. Generally $f(d, t) = 3.2d + 4.5t$, so $f(2, 3) = 19.9$

33. Jack is 6 years older than Jill. The sum of their ages is four times the age Jill was 8 years ago. How old are they?

Answer: $x - y = 6$ and $(x + y) = 4(y - 8)$ corresponds to $\begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -32 \end{bmatrix}$, which is solved by Jack $x = 25$ and Jill $y = 19$

34. You need three liters of a 30 percent alcohol solution. How can you mix 20 percent and 50 percent solutions to attain this?

Answer: $a \begin{bmatrix} .2 \\ .8 \end{bmatrix} + b \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} .3 \\ .7 \end{bmatrix}$ could be written $\begin{bmatrix} .2 & .5 \\ .8 & .5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} .3 \\ .7 \end{bmatrix}$ which is solved by $a = 2/3$ part of the 20% solution, and $b = 1/3$ part of the 50% solution

35. Fifty-four cookies are to be fed to ten pets, some dogs and some cats. Each dog gets 7 cookies, and each cat gets 3. How many dogs and cats are there?

Answer: $3c + 7d = 54$ and $c + d = 10$ could be written $\begin{bmatrix} 3 & 7 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 54 \\ 10 \end{bmatrix}$, and is solved by $c = 4$ and $d = 6$

36. A movie theater sells tickets for \$ 8. The student discount is \$ 2. One night 525 tickets were sold for a total revenue of \$ 3580. How many regular and student tickets were sold?

Answer: $r + s = 525$ and $8r + 6s = 3580$ is solved by $r = 215$ regular and $s = 310$ students

37. Your smoothie stand offers three smoothies using these fruit ingredients:

- **abracadabra:** \$ 3.50, 2 apples and 4 bananas
- **banana blast:** \$ 2.75, 6 bananas
- **citrus combo:** \$ 4.00, 1 apple, 2 bananas, 3 clementines

Each smoothie is 20 ounces. A customer's order is expressed as a vector of quantities q . Partial orders are allowed. e.g. $q_2 = 4.5$ means 90oz of banana blast.

(a) Write the vector p so that $p \cdot q$ is the total cost of the order.

Answer: $p = [3.5; 2.75; 4]$

(b) Write the matrix A such that Aq is how much of each fruit is required.

Answer: $A \begin{bmatrix} 2 & 0 & 1 \\ 4 & 6 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

(c) A particular order requires 18 apples, 48 bananas, and 12 clementines. Find the order vector q .

Answer: solve $Aq = \begin{bmatrix} 18 \\ 48 \\ 12 \end{bmatrix}$ to get $q = [7; 2; 4]$

(d) Find an order q that uses 10 apples, 25 bananas, and 15 clementines. How much does it cost?

Answer: $q = [2.5; 5/6; 5]$

(e) Find an order q that uses 10 apples, 15 bananas, and 20 clementines. Anything weird ?

Answer: $q = [5/3; -5/6; 20/3]$, needs negative bananas

38. Set up and solve a linear system that represents these chemical equations:

(a) $S_8 + F_2 \rightarrow SF_6$

Answer: 1,24,8

(b) $C_5H_{12} + O_2 \rightarrow CO_2 + H_2O$

Answer: 1,8,5,6

(c) $CO_2 + H_2O \rightarrow C_6H_{12}O_6 + O_2$ (photosynthesis)

Answer: 6,6,1,6

39. Set up (but do not solve) a matrix equation $Ax = b$ for a linear system you could use to find:

- (a) the equation of a quadratic $f(x) = a + bx + cx^2$ that passes through $(0, 3)$, $(4, 2)$, and $(9, 9)$.

Answer: gives $a = 3, a + 4b + 4c = 2, a + 9b + 81c = 9$, so

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & 16 \\ 1 & 9 & 81 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 9 \end{bmatrix}$$

- (b) the equation of a circle that passes through $(0, 3)$, $(4, 2)$, and $(9, 9)$.

Answer: $x^2 + y^2 + ax + by + c = 0$ gives $9 + 3b + c = 0, 16 + 4 + 4a + 2b + c = 0, 81 + 81 + 9a + 9b + c = 0,$

$$\begin{bmatrix} 0 & 3 & 1 \\ 4 & 2 & 1 \\ 9 & 9 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -9 \\ -20 \\ -162 \end{bmatrix}$$

40. Explain the pattern for setting up a linear system to find:

- (a) the equation of a quadratic that passes through three given points

Answer: a_1 is all one's, a_2 is the x values, a_3 is the x^2 values, RHS is the y vals

- (b) the equation of a circle that passes through three given points

Answer: a_1 is the x 's, a_2 is the y 's, a_3 is all ones, RHS is $-(x^2 + y^2)$ values

41. Is it true? Explain if there are caveats.

- (a) There is a unique quadratic (parabola) $f(x) = a + bx + cx^2$ that fits through three distinct random points.

Answer: true if no two points are on the same vertical line

- (b) There is a unique circle that fits through three distinct random points.

Answer: true if the three points are not co-linear (on the same line)

Gauss Elimination

42. Solve the quadratic and circle systems above by equation substitution methods.

Answer: $[3, -59/60, 11/60]$ so $f(x) = 3 - 59/60x + 11/60x^2$
 $[-73/11, -171/11, 414/11]$ so $x^2 + y^2 - 73/11x - 171/11y + 414/11$

43. Solve the quadratic and circle systems above by Gauss Elimination. Show all elementary row operations to reduce the systems from $[A|b]$ to $[I|x]$. Use fractions.

Answer: for the first system:

- $R_2 \leftarrow R_2 - R_1$
- $R_3 \leftarrow R_3 - R_1$
- $\frac{1}{4}R_2$
- $R_3 \leftarrow 9R_2$
- $\frac{1}{45}R_3$
- $R_2 \leftarrow 4R_3$

for the 2nd system:

- $R_1 \leftrightarrow R_2$ (you could also swap with row 3)
- $\frac{1}{4}R_1$
- $R_3 \leftarrow 9R_1$
- $\frac{1}{3}R_2$

- $R_3 = 9/2R_2$
- $\frac{-4}{11}R_3$
- $R_2 = 1/3R_3$
- $R_1 = 1/4R_3$
- $R_1 = 1/2R_2$

44. For each augmented matrix, write and perform the next two row operations of the Gauss elimination algorithm for putting the system in RREF.

(a) $\left[\begin{array}{cc|c} 1 & 7 & 3 \\ -4 & 2 & 5 \end{array} \right]$

(c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array} \right]$

(e) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 0 & 4 & 6 \\ 0 & 3 & 1 & 7 \end{array} \right]$

(b) $\left[\begin{array}{ccc|c} 1 & 7 & 3 & 3 \\ 0 & 2 & 3 & 9 \\ 0 & 8 & 5 & 4 \end{array} \right]$

(d) $\left[\begin{array}{ccc|c} 5 & 0 & 0 & 4 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 3 & 6 \end{array} \right]$

(f) $\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 4 & 6 & 7 \\ 0 & 0 & 1 & 7 & 9 \\ 0 & 3 & 2 & 3 & 4 \end{array} \right]$

45. Solve the triangular system by back-substitution without doing any additional row operations.

(a) $\left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 2 & 8 \end{array} \right]$

Answer: $[5/3; 1/2; 3]$

Answer: $[5; 1/2; 1/3; 1/4]$

Answer: $[-7; 4]$

(b) $\left[\begin{array}{ccc|c} 3 & 7 & -4 & 3 \\ 0 & -2 & 3 & 8 \\ 0 & 0 & \frac{1}{3} & 2 \end{array} \right]$

(d) $\left[\begin{array}{ccc|c} 0 & 0 & 2 & -8 \\ 0 & 1 & 2 & 4 \\ 6 & 2 & 3 & 0 \end{array} \right]$

(f) $\left[\begin{array}{cccc|c} 1 & 0 & 6 & 0 & 5 \\ 0 & 1 & 0 & 3 & 6 \\ 0 & 0 & 3 & 0 & 7 \\ 0 & 0 & 0 & 4 & 8 \end{array} \right]$

Answer: $[-2; 12; -4]$

Answer: $[-9; 0; 7/3; 2]$

Answer: $[-8/3; 5; 6]$

(c) $\left[\begin{array}{ccc|c} 3 & 0 & 0 & 5 \\ 0 & 8 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array} \right]$

(e) $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 5 \\ 1 & 2 & 0 & 0 & 6 \\ 1 & 2 & 3 & 0 & 7 \\ 1 & 2 & 3 & 4 & 8 \end{array} \right]$

46. This augmented matrix is equivalent to the given RREF.

$$\left[\begin{array}{ccc|c} 3 & 5 & 2 & b_1 \\ 1 & 8 & 3 & b_2 \\ 3 & -2 & 5 & b_3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -3 \end{array} \right]$$

Find the vector b .

Answer: multiply $b = Ax = [30; 41; -21]$

47. Show all EROs to put this system in RREF.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 23 \\ 1 & 3 & 5 & 37 \\ 3 & 5 & 8 & 63 \end{array} \right]$$

Answer: reduces to reveal solution $[3; -2; 8]$

48. Set up the augmented matrix representing a system you could solve to find the given curves. Then show all ERO's to reduce the system to RREF. Check your answer with Octave.

- (a) circle through $(1, 2)$, $(2, 0)$, and $(7, 1)$
- (b) parabola through $(1, 2)$, $(2, 0)$, and $(7, 1)$
- (c) cubic through $(0, 0)$, $(1, 2)$, $(2, 0)$, and $(7, 1)$

49. Use these three matrices to do the given computations, or say undefined.

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

- (a) $A + 2B + I$ **Answer:** $\begin{bmatrix} 6 & 13 \\ 9 & 5 \end{bmatrix}$ (d) $C + D$ **Answer:** undefined (g) $E + 4E$ **Answer:** same as $5E$
 (b) $A + 2B^T + I$ (e) $C + D^T$ (h) $E + 4I$
 (c) $3C$ (f) $\frac{1}{2}(A - A^T)$ (i) $E + E^T$

50. Find three different solutions to each of these linear systems (represented as augmented matrices)

(a) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ **Answer:** $[-2; 2; 1]$ $[-2; 1; 0]$
Answer: any multiple of $[5; -4; 1]$ (d) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right]$ **Answer:** $[-15; 1; 6]$
 (b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{array} \right]$ (c) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ **Answer:** any multiple of $[-17, 2, 6]$
Answer: any multiple of $[-19, 3, 6]$

51. Below is a “magic square”. Solve the homogeneous system $Ax = \vec{0}$.

$$A = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

Answer: only the trivial solution $x = \vec{0}$

52. Below is a “magic square”. Solve the homogeneous system $Ax = \vec{0}$ if $x_2 = 15$

$$A = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

Answer: do $rref(A)$ to find $x = [5; 15; -15; -5]$

53. If $D = ABC$, $D \in \mathbb{R}^{5 \times 3}$, C is square, and A has 2 columns, then what is the size of B ?

Answer: 2×3

54. Use these three matrices to do the given computations (by hand, but check with Octave), or say undefined.

$$A = \begin{bmatrix} 1 & 3 \\ 9 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 5 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 11 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

- | | | |
|--|--|-----------------|
| (a) AB | (h) $(A + B)^2$ | (o) CE |
| (b) BA | Answer: $\begin{bmatrix} 81 & 48 \\ 54 & 81 \end{bmatrix}$ | (p) DE |
| (c) $AB - BA$ | (i) $A^2 + 2AB + B^2$ | (q) EC |
| (d) $A^T A$ | Answer: $\begin{bmatrix} 36 & 40 \\ 63 & 126 \end{bmatrix}$ | (r) ED |
| (e) AA^T | (j) AI | (s) $ABCD$ |
| (f) A^2 | (k) IA | (t) $ABDC$ |
| Answer: $\begin{bmatrix} 28 & 9 \\ 27 & 31 \end{bmatrix}$ | (l) $(A + 3I)^2$ | (u) $A(B + DC)$ |
| (g) A^4 | (m) CD | (v) ECA |
| Answer: $\begin{bmatrix} 1027 & 531 \\ 1593 & 1204 \end{bmatrix}$ | (n) DC | (w) EAC |
| | | (x) CDE |

55. Let A be a square matrix. Which of these must be symmetric?

- | | | | |
|-------------|------------|---------------|---------------|
| (a) $A^T A$ | (b) AA^T | (c) $A + A^T$ | (d) $A - A^T$ |
|-------------|------------|---------------|---------------|

Answer: all but the last one will be symmetric

56. Let $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $y = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ be vectors. Compute the outer products xy^T and yx^T .

Answer: $xy^T = \begin{bmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{bmatrix}$ and $yx^T = \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \end{bmatrix}$

57. Express this matrix as an outer product xy^T . i.e. find x and y .

$$\begin{bmatrix} 2 & 6 & 4 \\ 3 & 9 & 6 \\ 7 & 21 & 14 \end{bmatrix}$$

Answer: one possibility is $x = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

58. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, and diagonal matrices $C = \text{diag}(5, 2)$ and $D = \text{diag}(2, 4, -1)$.

Compute (or say undefined):

- | | | |
|--|---|---|
| (a) AC | (c) CA | (e) CAD |
| Answer: undefined | Answer: $\begin{bmatrix} 5 & 10 & 15 \\ 8 & 10 & 12 \end{bmatrix}$ | Answer: $\begin{bmatrix} 10 & 40 & -15 \\ 16 & 40 & -12 \end{bmatrix}$ |
| (b) AD | (d) DA | (f) DAC |
| Answer: $\begin{bmatrix} 2 & 8 & -3 \\ 8 & 20 & -6 \end{bmatrix}$ | Answer: undefined | Answer: undefined |

59. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

- (a) Find a non-trivial solution to $Ax = \vec{0}$.
- (b) Find a diagonal matrix D such that all the columns of AD are unit vectors.
Answer: $\text{diag}(1/\sqrt{66}, 1/\sqrt{93}, 1/\sqrt{126})$
- (c) Find a diagonal matrix D such that all the rows of DA are unit vectors.
Answer: $\text{diag}(1/\sqrt{14}, 1/\sqrt{77}, 1/\sqrt{194})$
- (d) Find a diagonal matrix D such that all the columns of AD have sum of 72.
Answer: $\text{diag}(6, 4.8, 4)$
- (e) Find a diagonal matrix D such that all the rows of DA have sum of 72
Answer: $\text{diag}(12, 4.8, 3)$

60. Make up two diagonal matrices and multiply them together. Is the result diagonal ?

Answer: yes

61. Make up two upper triangular matrices and multiply them together. Is the result upper triangular ?

Answer: yes

62. Make up two lower triangular matrices and multiply them together. Is the result lower triangular ?

Answer: yes

63. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Compute $A^{20} = A^{16}A^4$ by hand.

Answer: Let $B = A^4 = (A^2)^2 = \begin{bmatrix} 199 & 290 \\ 435 & 634 \end{bmatrix}$ and $A^{16} = B^4 = (B^2)^2 = \begin{bmatrix} 115007491351 & 167615035490 \\ 251422553235 & 366430044586 \end{bmatrix}$

64. For each wealth transfer matrix (as described in class),

- Find $x^{(1)}$ and $x^{(2)}$ if the initial wealth is $x^{(0)} = 100(\vec{1})$.
- Solve the system $Ax = x$ to find the equilibrium distribution x . You may want to practice doing rref by hand, but check with Octave.

(a) $A = \begin{bmatrix} .3 & .9 \\ .7 & .1 \end{bmatrix}$

Answer: $x = [112.5; 87.5]$

(b) $A = \begin{bmatrix} .1 & .2 & 0 \\ .4 & 0 & .6 \\ .5 & .8 & .4 \end{bmatrix}$

Answer: $x = [24.32; 109.46; 166.22]$

65. For each linear transformation:

- Sketch the image of the e_i, e_j square (it will be a parallelogram).
- Write a 2×2 matrix that accomplishes the linear transformation.
- Check your answer at

<https://integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php>

(a) stretch horizontally by a factor of 3

(b) reflect across the y-axis

(c) rotate 90 degrees counter-clockwise

Answer: note how this is different than reflecting across y-axis

(d) rotate $\pi/5$ radians

(e) shear horizontally with a factor of 2

(f) shear vertically with a factor of $\frac{1}{2}$

66. For each matrix representing a linear transformation:

- Sketch the image of the e_i, e_j square (it will be a parallelogram).
- Describe the geometric effect in words.
- Check your answer at

<https://integral-domain.org/lwilliams/Applets/algebra/linearTransformations.php>

(a) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

Answer: stretch both directions

(c) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Answer: reflect across x-axis

(e) $\begin{bmatrix} .5 & .866 \\ -.866 & .5 \end{bmatrix}$

Answer: rotate clockwise 60 degrees

(b) $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Answer: shear vertically

(d) $\begin{bmatrix} .866 & -.5 \\ .5 & .866 \end{bmatrix}$

Answer: rotate counter-clockwise 30 degrees

(f) $\begin{bmatrix} -.5 & -.866 \\ .866 & -.5 \end{bmatrix}$

Answer: rotate counter-clockwise 120 degrees

67. Let $R = \begin{bmatrix} .75 & * \\ * & * \end{bmatrix}$ be a rotation matrix. What are the two possibilities for R ?

Answer: rotate $\pm \cos^{-1}(.75)$, so $R = \begin{bmatrix} .75 & -.6614 \\ .6614 & .75 \end{bmatrix}$ or $R = \begin{bmatrix} .75 & .6614 \\ -.6614 & .75 \end{bmatrix}$

68. Write out and perform matrix multiplication to find the matrix representing these combinations of linear transformations. Indicate which ones work out to be the same thing regardless of which order they are applied (i.e. the matrices commute)

(a) stretch by factor of 5 horizontally, then stretch by a factor of 8 vertically

Answer: $\begin{bmatrix} 5 & 0 \\ 0 & 8 \end{bmatrix}$ commutative

(b) rotate 30 degrees counter-clockwise, then rotate 90 degrees counter-clockwise

Answer: net rotation of 120 degrees, commutative

(c) rotate 30 degrees counter-clockwise, then reflect across the x-axis

Answer: not commutative

(d) rotate 30 degrees counter-clockwise, then stretch by a factor of 3 horizontally

Answer: not commutative

- (e) rotate 30 degrees counter-clockwise, then shear horizontally with a factor of 2
Answer: not commutative
- (f) shear horizontally with a factor of 2, then shear vertically with a factor of 3
Answer: not commutative
- (g) shear horizontally with a factor of 2, then stretch by a factor of 3 vertically
Answer: not commutative
- (h) shear horizontally with a factor of 2, then reflect across the x-axis
Answer: not commutative
- (i) reflect across the x-axis, then reflect across the y-axis
Answer: commutative

69. Multiply matrices to accomplish the given sequence (you may use Octave)

- (a) rotate 30 degrees counter-clockwise, then reflect across the x-axis, then shear vertically with a factor of 3
Answer: $\begin{bmatrix} .866 & -.5 \\ 2.1 & -2.37 \end{bmatrix}$
- (b) stretch by a factor of 5 horizontally, then rotate 30 degrees counter-clockwise, then shear horizontally with a factor of 2, then reflect across the y-axis.
Answer: $\begin{bmatrix} -9.33 & -1.23 \\ 2.5 & .866 \end{bmatrix}$

70. What is the net effect of:

- (a) stretch by a factor of 2 horizontally, then stretch by a factor of $\frac{1}{2}$ horizontally
- (b) rotate 30 degrees counter-clockwise, then rotate 30 degrees clockwise
- (c) reflect across the x-axis, then reflect across the x-axis
- (d) shear horizontally with a factor of 2, then shear horizontally with a factor of -2

Answer: all of these compositions undo each other, with the net effect being the identity matrix

71. Let A be a 5×5 matrix with $a_{ij} = 5(i - 1) + j$. Write/evaluate the following:

(a) A

$$\text{Answer: } \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{bmatrix}$$

(b) Ae_3

$$\text{Answer: } \begin{bmatrix} 3 \\ 8 \\ 13 \\ 18 \\ 23 \end{bmatrix}$$

(c) $A^T e_3$

$$\text{Answer: } \begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \\ 15 \end{bmatrix}$$

(d) $e_2^T A$

$$\text{Answer: } [6, 7, 8, 9, 10]$$

(e) $e_2^T A^T$

$$\text{Answer: } [2, 7, 12, 17, 22]$$

(f) $e_2^T A e_3$

$$\text{Answer: } 8$$

(g) $e_2^T A^T e_3$

$$\text{Answer: } 12$$

(h) $e_2^T A \vec{1}$

$$\text{Answer: } 6 + 7 + 8 + 9 + 10 = 40$$

(i) $\vec{1}^T A e_2$

$$\text{Answer: } 2 + 7 + 12 + 17 + 22 = 60$$

72. Find the following:

(a) rotation $R = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$, find R^{-1}

$$\text{Answer: } R^{-1} = \begin{bmatrix} .96 & .28 \\ -.28 & .96 \end{bmatrix}$$

(b) scaling $D = \begin{bmatrix} .25 & 0 \\ 0 & 1.25 \end{bmatrix}$, find D^{-1}

$$\text{Answer: } D^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & .8 \end{bmatrix}$$

(c) shear $S = \begin{bmatrix} 1 & 0 \\ .75 & 1 \end{bmatrix}$, find S^{-1}

$$\text{Answer: } S^{-1} = \begin{bmatrix} 1 & 0 \\ -.75 & 1 \end{bmatrix}$$

(d) the composition $T = DR$, find T and T^{-1}

(e) the composition $T = RD$, find T and T^{-1}

(f) the composition $T = RS$, find T and T^{-1}

(g) the composition $T = RDS$, find T and T^{-1}

73. Compute $\det(A)$, and if possible, find A^{-1} .

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\text{Answer: } A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\text{Answer: } A^{-1} = \frac{1}{1} \begin{bmatrix} 4 & -5 \\ -7 & 9 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 9 & -5 \\ 7 & 4 \end{bmatrix}$

$$\text{Answer: } A^{-1} = \frac{1}{71} \begin{bmatrix} 4 & 5 \\ -7 & 9 \end{bmatrix}$$

(d) $A = \begin{bmatrix} 9 & 3 \\ 6 & 2 \end{bmatrix}$ **Answer:** $A^{-1} = \frac{1}{-15} \begin{bmatrix} 0 & -3 \\ -5 & 0 \end{bmatrix}$

Answer: $\det(A) = 0$, so not invertible

(f) $A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

Answer: $A^{-1} = \frac{1}{-4} \begin{bmatrix} -2 & -1 \\ -2 & 1 \end{bmatrix}$

(e) $A = \begin{bmatrix} 0 & 3 \\ 5 & 0 \end{bmatrix}$

74. The matrix $A = \begin{bmatrix} x & 5 \\ x-2 & 8 \end{bmatrix}$ is not invertible. Find the value of $x \in \mathbb{R}$.

Answer: set $\det(A) = 0$ to get $8x - 5(x - 2) = 0$, so $x = -10/3$

75. Let $A^{-1} = \begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix}$.

(a) If $Ax = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$, then find $x \in \mathbb{R}^2$.

Answer: $A^{-1} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 26/8 \\ 20/8 \end{bmatrix}$

(b) Is it true that $(A^{-1})^{-1} = A$?

Answer: yes

(c) Compute $A^2 e_2$.

Answer: $A = \frac{1}{8} \begin{bmatrix} 6 & -5 \\ -2 & 3 \end{bmatrix}$, $A(Ae_2) = A \begin{bmatrix} -5 \\ 3 \end{bmatrix} / 8 = \begin{bmatrix} -45/64 \\ 19/64 \end{bmatrix}$

76. If $A^2 = I$, then A is its own inverse. Find three examples of 2×2 matrices with this property.

77. Let $A = \begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix}$. Verify that $(A^T)^{-1} = (A^{-1})^T$.

78. Let $A = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The matrix $M \in \mathbb{R}^{2 \times 2}$ is a mystery matrix. Solve for M if:

(a) $AM = I$

Answer: $M = A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

(f) $BM = A$

Answer: $M = B^{-1}A$

(b) $MA = I$

Answer: $M = A^{-1}$

(g) $AMB = I$

Answer: $M = A^{-1}B^{-1}$

(c) $AM = B$

Answer: $M = A^{-1}B$

(h) $AMB = A$

Answer: $M = B^{-1}$

(d) $MA = B$

Answer: $M = BA^{-1}$

(i) $AMA = A$

Answer: $M = A^{-1}$

(e) $AM = A$

Answer: $M = I$

(j) $AMA = B$

Answer: $M = A^{-1}BA^{-1}$

79. Let $A = \begin{bmatrix} 8 & 3 \\ 5 & 2 \end{bmatrix}$. Compute A^k for $k = -3, -2, -1, 0, 1, 2, 4, 8$

Answer: $A^{-1} = \begin{bmatrix} 2 & -3 \\ -5 & 8 \end{bmatrix}$, then e.g. $A^{-3} = (A^{-1})^3$, $A^8 = (A^4)^2 = ((A^2)^2)^2$

80. Let A be a square matrix. If $M^2 = A$, we may write $M = \sqrt{A}$. Find square roots of these matrices:

(a) rotation $R = \begin{bmatrix} .96 & -.28 \\ .28 & .96 \end{bmatrix}$

Answer: the angle of rotation is $\tan^{-1} .28/.96 = 16.26^\circ$, so we could rotate by 8.13° twice to obtain the same effect. $M = \begin{bmatrix} .98995 & -.14142 \\ .14142 & .98995 \end{bmatrix}$

(b) scaling $D = \begin{bmatrix} .25 & 0 \\ 0 & 1.25 \end{bmatrix}$

Answer: $M = \begin{bmatrix} \sqrt{.25} & 0 \\ 0 & \sqrt{1.25} \end{bmatrix}$

(c) shear $S = \begin{bmatrix} 1 & 0 \\ .75 & 1 \end{bmatrix}$

Answer: we could shear by half as much twice: $M = \begin{bmatrix} 1 & 0 \\ .75/2 & 1 \end{bmatrix}$

81. Wealth transfers between two entities according to this model:

$$x^{(k+1)} = Ax^{(k)}$$

If $A = \begin{bmatrix} .2 & .7 \\ .8 & .3 \end{bmatrix}$, and $x^{(3)} = \begin{bmatrix} 52 \\ 48 \end{bmatrix}$, find $x^{(0)}$.

Answer: $A^{-3}x^{(3)} = \begin{bmatrix} 4 \\ 96 \end{bmatrix}$

82. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. Compute this ratio for large n . Can you guess what number it approaches ?

$$\frac{e_1 A^n e_1}{e_2^T A^n e_1}$$

Answer: $1 + \sqrt{2}$, which is called the silver ratio

83. Let $A = \begin{bmatrix} 8 & 32 \\ 15 & 50 \end{bmatrix}$.

(a) Find a rotation R so that $RA = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

Answer: let $\theta = -\tan^{-1}(15/8)$ so the rotation matrix is $R = \begin{bmatrix} .471 & .882 \\ -.882 & .471 \end{bmatrix}$, and $RA = \begin{bmatrix} 17 & 59.2 \\ 0 & -4.71 \end{bmatrix}$

(b) Find a diagonal D so that $RAD = \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$

Answer: $D = \begin{bmatrix} 1/17 & 0 \\ 0 & -1/4.71 \end{bmatrix}$ and $RAD = \begin{bmatrix} 1 & -12.6 \\ 0 & 1 \end{bmatrix}$

(c) Find a shear S so that $SRAD = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer: $S = \begin{bmatrix} 1 & 12.6 \\ 0 & 1 \end{bmatrix}$

(d) Factor A as a product of three matrices: $A = R^{-1}S^{-1}D^{-1}$.

Answer: $A = \begin{bmatrix} .471 & -.882 \\ .882 & .471 \end{bmatrix} \begin{bmatrix} 1 & -12.6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 17 & 0 \\ 0 & -4.71 \end{bmatrix}$

(e) Verify that $A^{-1} = DSR$

84. Let $A = \begin{bmatrix} 8 & 32 \\ 15 & 50 \end{bmatrix}$.

(a) Find a shear S_1 so that $S_1A = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$

Answer: $S_1 = \begin{bmatrix} 1 & 0 \\ -15/8 & 1 \end{bmatrix}$ and $S_1A = \begin{bmatrix} 8 & 32 \\ 0 & -10 \end{bmatrix}$

(b) Find a shear S_2 so that $S_2S_1A = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$

Answer: $S_2 = \begin{bmatrix} 1 & 32/10 \\ 0 & 1 \end{bmatrix}$ and $S_2S_1A = \begin{bmatrix} 8 & 0 \\ 0 & -10 \end{bmatrix}$

(c) Find a diagonal D so that $DS_2S_1A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Answer: $D = \begin{bmatrix} 1/8 & 0 \\ 0 & -1/10 \end{bmatrix}$

(d) Verify that $A^{-1} = DS_2S_1$

(e) Factor A as a product of three matrices.

Answer: $A = S_1^{-1}S_2^{-1}D^{-1} = \begin{bmatrix} 1 & 0 \\ 15/8 & 1 \end{bmatrix} \begin{bmatrix} 1 & -32/10 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & -10 \end{bmatrix}$

85. Experiment with 2×2 matrices to find a counter-example to each of these statements.

(a) If A and B are invertible, then $(A + B)^{-1} = A^{-1} + B^{-1}$

Answer: let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $(A + B)^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} .5 & -.25 \\ 0 & .5 \end{bmatrix}$ but
 $A^{-1} + B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$

(b) If $AB = 0$, then either $A = 0$ or $B = 0$.

Answer: $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) If $AB = AC$, then $B = C$.

Answer: let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $AB = AC = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$

(d) If $Ax = 0$ then $A^T x = 0$.

Answer: let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, then $A \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but $A^T \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

86. For each matrix, if possible:

- Factor as $A = QR$, where $Q^T Q = I$ and R is upper triangular.
- Factor as $A = LU$, where L is lower triangular, and U is upper triangular.

(a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Answer: $A = \begin{bmatrix} .316 & -.949 \\ .949 & .316 \end{bmatrix} \begin{bmatrix} 3.16 & 4.43 \\ 0 & -.632 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

(b) $A = \begin{bmatrix} -3 & 1 \\ 5 & 7 \end{bmatrix}$

Answer: $A = \begin{bmatrix} .5145 & .8575 \\ -.8575 & .5145 \end{bmatrix} \begin{bmatrix} -5.83 & -5.49 \\ 0 & 4.46 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -5/3 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 0 & 26/3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}$

Answer: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 8 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & 8 \\ 2 & 4 \end{bmatrix}$

Answer: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & -8 \end{bmatrix}$; LU not possible

87. Let $A = \begin{bmatrix} 3 & -12 \\ 2 & 7 \end{bmatrix}$. This sequence of row operations produces the equivalent RREF:

- divide row 1 by 3
- subtract 2 times row 1 from row 2
- scale row 2 by 1/15
- add 4 times row 2 to row 1

(a) Write each row operation as a matrix multiplication, so that $M_4 M_3 M_2 M_1 A = I$.

Answer: $\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/15 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -12 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) Write A^{-1} as the product of four matrices.

Answer: $A^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/15 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}$

(c) Write A as the product of four matrices.

Answer: $A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

88. Find the angle between the vectors $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

Answer: $\cos^{-1}\left(\frac{v^T x}{\|v\| \|x\|}\right) = .103 = 5.91^\circ$

89. Find two values of y so that the angle between $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ y \end{bmatrix}$ is 30° .

Answer: solve $\frac{20+3y}{5\sqrt{25+y^2}} = \cos 30 = \sqrt{3}/2$; this works out to be a quadratic with solutions $y \approx .602, 11.7$

90. Find the angle between the vectors $\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ -6 \end{bmatrix}$.

Answer: $\cos^{-1}\left(\frac{v^T x}{\|v\| \|x\|}\right) = 67.8^\circ$

91. Find the angle between the lines $y = 2x$ and $y = 5 - 3x$.

Answer: let $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, then $\cos^{-1}\left(\frac{v^T x}{\|v\| \|x\|}\right) = 135^\circ$ (the acute angle would be 45 degrees)

92. For each vector v ,

- Find the matrix P for the orthogonal projection onto v .
- Find the reflection matrix R for the orthogonal reflection across v .

(a) $v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

Answer: $P = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$ and $R = \begin{bmatrix} -.28 & .96 \\ .96 & .28 \end{bmatrix}$

(b) $v = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

Answer: same answer as for $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

(c) $v = \begin{bmatrix} 45 \\ -28 \end{bmatrix}$

Answer: $P = \begin{bmatrix} .721 & -.449 \\ -.449 & .279 \end{bmatrix}$ and $R = \begin{bmatrix} .442 & -.897 \\ -.897 & -.442 \end{bmatrix}$

(d) $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Answer: $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(e) $v = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

Answer: $P = \begin{bmatrix} .0476 & .0952 & .1905 \\ .0952 & .1905 & .381 \\ .1905 & .381 & .762 \end{bmatrix}$ and $R = \begin{bmatrix} -.905 & .1905 & .381 \\ .1905 & -.619 & .762 \\ .381 & .762 & .524 \end{bmatrix}$

(f) $v = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Answer: $P = \begin{bmatrix} .2 & 0 & .4 \\ 0 & 0 & 0 \\ .4 & 0 & .8 \end{bmatrix}$ and $R = \begin{bmatrix} -.6 & 0 & .8 \\ 0 & -1 & 0 \\ .8 & 0 & .6 \end{bmatrix}$

93. Find the matrix P that orthogonally projects vectors onto the line $y = \frac{7}{3}x$.

Answer: $P = \begin{bmatrix} .155 & .362 \\ .362 & .845 \end{bmatrix}$

94. Find the matrix R that orthogonally reflects vectors across the line $y = \frac{7}{3}x$.

Answer: $R = 2P - I = \begin{bmatrix} -.69 & .724 \\ .724 & .69 \end{bmatrix}$

95. Find the matrix that represents the composition:

(a) first rotate 45° , then project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

Answer: $R = \begin{bmatrix} .707 & -.707 \\ .707 & .707 \end{bmatrix}$ and $P = \begin{bmatrix} .221 & .415 \\ .415 & .779 \end{bmatrix}$; the composition is $PR = \begin{bmatrix} .450 & .137 \\ .844 & .257 \end{bmatrix}$

(b) first project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$, then rotate 45°

Answer: the composition is $RP = \begin{bmatrix} -.137 & -.257 \\ .450 & .844 \end{bmatrix}$

(c) first scale by $\text{diag}(3, 2)$, then project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$

Answer: $D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$, and the composition is $PD = \begin{bmatrix} .664 & .830 \\ 1.25 & 1.56 \end{bmatrix}$

(d) first project onto $\begin{bmatrix} 8 \\ 15 \end{bmatrix}$, then scale by $\text{diag}(3, 2)$

Answer: $DP = \begin{bmatrix} .664 & 1.25 \\ .830 & 1.56 \end{bmatrix}$

96. Given an orthogonal projection $P = \frac{vv^T}{v^T v}$ onto v , let $F(\alpha) = (1 - \alpha)P + \alpha I$ be the corresponding flattening operator.

(a) Find the inverse of $F(0.5)$.

Answer: $F(2)$

(b) Find the inverse of $F(0.2)$.

Answer: $F(5)$

(c) Find the inverse of $F(-1)$.

Answer: $F(-1)$

generally $F(\alpha)F(1/\alpha) = I$

97. Let $x \in \mathbb{R}^n$, and $\vec{1} \in \mathbb{R}^n$ be the vector of all ones.

(a) Write the projection matrix P onto $\vec{1}$.

Answer: let $v = \vec{1}$, so that $v^T v = n$. Then the projection is $P = \frac{1}{n} \vec{1} \vec{1}^T$. e.g. if $n = 3$ then

$$P = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(b) If $x = \begin{bmatrix} 5 \\ 9 \\ 6 \\ 4 \end{bmatrix}$, find Px . Can you describe what Px is doing?

Answer: subtracts the mean

(c) Show algebraically that $1^T Px = 0$ for all vectors x .

Answer: $1^T (I - \frac{1}{n} 11^T)x = 0$

98. Find the determinant of each matrix by hand (you can either row reduce to an upper triangular, or use cofactor expansion)

(a) $[5]$

Answer: 5

(b) $\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$

Answer: 7

(c) $\begin{bmatrix} 5 & 3 & 1 \\ 1 & 2 & 4 \\ 4 & 3 & -2 \end{bmatrix}$

Answer: -31

(d) $\begin{bmatrix} 5 & 3 & 1 & 3 \\ 1 & 2 & 4 & 8 \\ 4 & 3 & -2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$

Answer: 46

(e) $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 5 & 0 \\ 8 & 2 & 3 \end{bmatrix}$

Answer: 30

(f) $\begin{bmatrix} 8 & 2 & 3 \\ 1 & 5 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

Answer: -30

(g) $\begin{bmatrix} 2 & 3 & 0 & 2 \\ 2 & 6 & -1 & 5 \\ 4 & 12 & -3 & 7 \\ 2 & 3 & -1 & -1 \end{bmatrix}$

Answer: 0

(h) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

Answer: -7

(i) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{bmatrix}$

Answer: 5

(j) $\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}$

Answer: 33

(k) $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

Answer: 0

(l) $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$

Answer: 1

99. Is the statement generally true or false ? (assume A and B are $n \times n$ matrices).

(a) $\det(AB) = \det(BA)$

Answer: true

(b) If A is singular, then $\det(A) = 0$.

Answer: true

(c) $\det(A + B) = \det(A) + \det(B)$

Answer: false

(d) $\det(-A) = -\det(A)$

Answer: false, multiply by $(-1)^n$, which is positive if n is even

(e) $\det(A^{-1})\det(A) = 1$ if A is invertible

Answer: true

(f) $\det(A^T A) = \det(A)^2$

Answer: true, $\det(A^T) = \det(A)$

(g) $\det(3A) = 3^n \det(A)$

Answer: true

(h) $\det(A + 2I) = \det(A) + 2$

Answer: false

- (i) If you swap two rows of A , the determinant doesn't change.
Answer: false, multiply by -1
- (j) If rows of A satisfy $a_1 + a_2 = a_3$, then $\det(A) = 0$.
Answer: true
- (k) If columns of A satisfy $a_1 + a_2 = a_3$, then $\det(A) = 0$.
Answer: true
- (l) If $A^T = -A$ and n is odd, then $\det(A) = 0$.
Answer: true, since $\det(A) = \det(A^T) = \det(-A) = -\det(A)$

100. Find all possible values of x .

(a) $\det \begin{bmatrix} 3 & 4 \\ 2 & x \end{bmatrix} = 12$

Answer: $3x - 8 = 12$, so $x = 20/3$

(b) $\det \begin{bmatrix} x & 4 \\ 2 & x \end{bmatrix} = 12$

Answer: $x^2 - 8 = 12$, so $x = \sqrt{20}$

(c) $\det \begin{bmatrix} x & 4 \\ x & x \end{bmatrix} = 12$

Answer: $x^2 - 4x = 12$, so $x = 6, -2$

(d) $\det \begin{bmatrix} x & 4 \\ x & x \end{bmatrix} = x$

Answer: $x^2 - 4x = x$, so $x = 0, 5$

(e) $\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \det \begin{bmatrix} 1 & 0 \\ 1 & x \end{bmatrix} = \det \begin{bmatrix} 2 & 2 \\ 4 & 4+x \end{bmatrix}$

Answer: $x = -1$

(f) $\det \left(\begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix} + xI \right) = 0$

Answer: $(x+1)(x+4) - 18 = 0$ gives $x = 2, -7$

(g) $\det \left(\begin{bmatrix} 1 & 2 \\ 9 & 4 \end{bmatrix} + xI \right) = 12$

Answer: $(x+1)(x+4) - 18 = 12$ gives $x = \frac{-5 \pm \sqrt{129}}{2}$

(h) $\det \begin{bmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{bmatrix} = 0$

Answer: expand across the top row to get $1 - x^2 - x^2 = 0$ so $x = \pm\sqrt{1/2}$

(i) $\det \begin{bmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{bmatrix} = 0$

Answer: expand across the top row to get $x(x^2 - 1) - x = 0$ so $x = 0, \pm\sqrt{2}$

101. Find the adjoint of these matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Answer: $\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Answer: $\begin{bmatrix} 24 & -12 & -2 \\ 0 & 6 & -5 \\ 0 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Answer: $\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

Answer: $\begin{bmatrix} 24 & -12 & -2 & 0 \\ 0 & 6 & -5 & 0 \\ 0 & 0 & 4 & 0 \\ -24 & 6 & 3 & 24 \end{bmatrix}$

102. If A has all integer entries, and $\det(A) = \pm 1$, then A^{-1} has all integer entries. To produce such a matrix, start with a triangular matrix with 1's on the Gordon line, and do random row operations (using integers). Let's illustrate:

(a) Start with $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Do some row operations, and write the resulting matrix A .

- $R_2 += 3R_1$
- $R_3 -= 2R_2$
- swap rows 1 and 3
- $R_3 -= R_1$

(c) Check that $\det(A)$ is either 1 or -1.

Answer: $\det \begin{bmatrix} -6 & -14 & -25 \\ 3 & 7 & 13 \\ 7 & 16 & 28 \end{bmatrix} = -1$ (since there was a swap)

(d) Find A^{-1} using the adjoint formula.

Answer: $A^{-1} = \begin{bmatrix} 12 & 8 & 7 \\ -7 & -7 & -3 \\ 1 & 2 & 0 \end{bmatrix}$

103. Create a 4×4 matrix A of integers, with no zero entries, having $\det(A) = 20$.

Hint: start with a triangular matrix with the desired determinant, then do row operations to fill in the zeros.

104. Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$.

(a) Find $u \times v$.

Answer: $\begin{bmatrix} -10 \\ 17 \\ -8 \end{bmatrix}$

(b) Find $v \times u$.

Answer: $\begin{bmatrix} 10 \\ -17 \\ 8 \end{bmatrix}$

(c) Find $(2u) \times v$.

Answer: $\begin{bmatrix} -20 \\ 34 \\ -16 \end{bmatrix}$

(d) Find $(u + v) \times v$.

Answer: $\begin{bmatrix} -10 \\ 17 \\ -8 \end{bmatrix}$

(e) Find $u \times u$.

Answer: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

105. Find the area of the triangles with given corners.

(a) $P(0, 0)$, $Q(-2, 9)$, $R(5, 3)$

Answer: $|\det \begin{bmatrix} -2 & 5 \\ 9 & 3 \end{bmatrix}| = 51/2$

(b) $P(8, 2)$, $Q(15, 5)$, $R(7, 10)$

Answer: $|\det \begin{bmatrix} 8 & 15 & 7 \\ 2 & 5 & 10 \\ 1 & 1 & 1 \end{bmatrix}| = 59/2$

(c) $P(0, 0, 0)$, $Q(-2, 9, 4)$, $R(5, 3, 12)$

Answer: $\frac{1}{2} \|\vec{PQ} \times \vec{PR}\| = \frac{1}{2} \sqrt{13753}$

(d) $P(8, 2, 3)$, $Q(15, 5, 11)$, $R(7, 10, 6)$

Answer: $\frac{1}{2} \|\vec{PQ} \times \vec{PR}\| = \frac{1}{2} \sqrt{7347}$

106. The volume of a triangular pyramid (also known as a tetrahedron) is $\frac{1}{6} |\det(A)|$ where the columns of A are the edges emanating from one corner (which we can place at the origin). Find the volume of

such a pyramid if the edges are $u = \begin{bmatrix} 10 \\ 6 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ 12 \\ 3 \end{bmatrix}$, and $w = \begin{bmatrix} 5 \\ 5 \\ 20 \end{bmatrix}$.

Answer: $\frac{1}{6} \det[u, v, w] = 390$

107. Sketch the pentagon with corners at $(0, 0)$, $(-1, 3)$, $(4, 6)$, $(5, 2)$, and $(4, 0)$. Divide it into 3 triangles, and find the area of the pentagon.

108. Let $u = \begin{bmatrix} x \\ 5 \end{bmatrix}$ and $v = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$. Find the two values of x so that the parallelogram determined by u and v has area 100.

Answer: $|\det \begin{bmatrix} x & 7 \\ 5 & 3 \end{bmatrix}| = 100$, so $|3x - 35| = 100$ so $x = 135/3$ or $x = -65/3$

109. Find the volume of the parallelepiped determined by the columns of $A = \begin{bmatrix} 4 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$.

Answer: $\det(A) = 14$

110. Let $A = \begin{bmatrix} 1 & x & 0 \\ x & 1 & x \\ 0 & x & 1 \end{bmatrix}$ determine a parallelepiped.

(a) Find the volume if $x = 0$.

Answer: 1

(b) Find the volume if $x = 1$.

Answer: 1

(c) Find the volume if $x = 2$.

Answer: 7

(d) Find the volume if $x = 3$.

(e) Determine the value of x such that the volume is 100.

111. Let $u = \begin{bmatrix} 6 \\ 17 \\ 10 \end{bmatrix}$ and $v = \begin{bmatrix} 21 \\ 5 \\ 32 \end{bmatrix}$. Find a vector w that is orthogonal to both u and v , scaled so that $\|w\| = 50$.

Answer: $u \times v = \begin{bmatrix} 494 \\ 18 \\ -327 \end{bmatrix}$ has length $\sqrt{351289}$, so let $w = \frac{50}{\sqrt{351289}} \begin{bmatrix} 494 \\ 18 \\ -327 \end{bmatrix} \approx \begin{bmatrix} 41.674 \\ 1.5185 \\ -27.5858 \end{bmatrix}$

112. Find a value of λ such that $\det \begin{bmatrix} 4 - \lambda & 6 \\ 6 & 4 - \lambda \end{bmatrix} = 0$

Answer: solve $(4 - \lambda)^2 = 36$ to get $\lambda = -2, 10$

113. Let $v \in \mathbb{R}^n$ (with $n > 1$) be any unit vector. Then $P = vv^T$ is a matrix that projects onto v . Using an example, argue that $\det(P) = 0$.

Answer: e.g. $v = \begin{bmatrix} .6 \\ .8 \end{bmatrix}$, then $P = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$. The rows are scalar multiples of each other, so e.g. $R_2 - \frac{4}{3}R_1$ yields a row of zeros, hence the determinant is zero.

114. Suppose $Q^T Q = I$. Argue that $\det(Q) = \pm 1$.

Answer: $1 = \det(I) = \det(Q^T Q) = \det(Q^T) \det(Q) = \det(Q) \det(Q) = \det(Q)^2$, so $\det(Q)$ is either 1 or -1.