1 C-N Math 208 - Spring 2020

1. linear: change in output proportional to change in input
   (flat/straight - most things are if you zoom in)

2. with proper choice of origin
   additivity $f(x+y) = f(x) + f(y)$ and homogeneity $f(cx) = cf(x)$, e.g. cost of 5 apples

3. "non-linear", e.g diminishing returns, energy $m|v|^2$
   little wine good for the stomach, but not too much

4. algebra (use rules to assemple pieces; puzzle)

5. vector: array/ordered list of scalars
   - $v \in \mathbb{R}^3$ means a vector with three real number components
   - dimension, subscripts, transpose, dot product
   - scalar multiplication, addition, linear combination
   - fruit purchases, cost is prices times quantities $p \cdot q$
   - same total bill if pay seprately: distributive $p \cdot (q_A + q_B) = p \cdot q_A + p \cdot q_B$
   - convert EUR-USD before or after: associative $(.85p) \cdot q = .85(p \cdot q)$

6. Linear function $f(x_1, x_2)$ with $f(5,0) = 45$ and $f(2,4) = 32$, find $f(1,3)$.
   - e.g. prices of different 'baskets' (assuming $f(0,0) = 0$)
   - $f(x_1, x_2) = 9x_1 + 3.5x_2 = \begin{bmatrix} 9 \\ 3.5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ can be expressed as a dot product

7. matrix: dimensions, row/column indexing, transpose, Octave notation

8. geometric view of vectors, illustrate with $v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$
   - draw as arrow with initial and terminal points
   - length/magnitude/norm: $\|v\| = \sqrt{v \cdot v} = \sqrt{3^2 + 1^2} = \sqrt{10}$
   - scalar multiple, e.g. $-2v = \begin{bmatrix} -6 \\ -2 \end{bmatrix}$ (stretched and flipped)
   - $\|cv\| = |c|\|v\|$
   - unit vectors have length 1. $\frac{v}{\|v\|} = \frac{1}{\|v\|}v$ is a unit vector
   - addition, e.g. $v + w = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$, draw "tail to head" and look at net result
   - linear combination, e.g. $2v - 3w = \begin{bmatrix} 0 \\ 17 \end{bmatrix}$
   - triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$
   - given vectors representing points, the vector from P to Q is $\vec{PQ} = Q - P$

9. linear combinations as matrix-vector products

- 3 types of smoothies, each with different fruit (apple,banana,clementine,dragonfruit) ingredients $S = \begin{bmatrix} 3 & 1 & 5 \\ 4 & 2 & 0 \\ 2 & 6 & 4 \\ 1 & 3 & 4.5 \end{bmatrix}$, and a vector $x = \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix}$ of coefficients.
  $Sx = 5s_1 + 8s_2 + 2s_3$ gives the amount of each fruit required for smoothie order
- In general, Ax gives a linear combination of the columns of A. If $A \in \mathbb{R}^{m\times n}$, then $x \in \mathbb{R}^n$, and $Ax \in \mathbb{R}^m$.
- To get a linear combination of the rows of A, pre-multiply by a row vector of coefficients. e.g. if $A \in \mathbb{R}^{4\times 7}$ then $2a_{1:} + 5a_{3:} - \pi a_{4:} = [2, 0, 5, -\pi]A$. The result is a 7 dimensional row vector.

10. orthogonality
    - perpendicular lines have slopes that are negative reciprocals (rise/run is vectorized as $\begin{bmatrix} run \\ rise \end{bmatrix}$)
    - vectors are orthogonal (perpendicular, meet at a right angle), $v \perp w$, if and only if $v \cdot w = 0$
    - a consequence of the Pythag. theorem:
      $\|v\|^2 + \|w\|^2 = \|v + w\|^2$ written as $v \cdot v + w \cdot w = (v + w) \cdot (v + w)$.
11. linear systems in $\mathbb{R}^2$
    - intersection of two lines, e.g. $y = 2 + x$ and $y = 3x - 8$
    - linear combination of two vectors, $5\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 7\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$
    - matrix equation $Ax = b$ (two dot product conditions), $\begin{bmatrix} -1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$
      b is called the "right-hand side (RHS)"
12. solving simple systems (easy to check)
    - diagonal matrix (in 2D, intersection of horizontal and vertical lines)
    - triangular matrix (back-substitute)
    - 2×2, add multiple of one equation to the other to eliminate a variable
13. applications (see HW)
    - prices and quantities
    - mixing solutions
    - riddles
14. balancing chemical equations `https://www.scirp.org/pdf/ALAMT_2016121515205487.pdf`
    (a) iron and oxygen to rust: $Fe + O_2 \to Fe_2O_3$. vectors represent the atomic ingredients

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 2 & -3 \end{bmatrix} x = \vec{0}$$

solutions are multiples of $[4; 3; 2]$

(b) $C_3H_8 + O_2 \to H_2O + CO_2$

$$\begin{bmatrix} 3 & 0 & 0 & -1 \\ 8 & 0 & -2 & 0 \\ 0 & 2 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

solve **homogeneous** system by letting $x_4 = 1$. Scale solution to get all integers.

(c) four types of smoothies using four fruits, or four molecules using four atoms
$NaOH + H_2SO_4 \to Na_2SO_4 + H_2O$

$$\begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 2 \\ 0 & 1 \end{bmatrix} x = \begin{bmatrix} 2 & 0 \\ 4 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} y$$

is solved by multiples of $x = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $y = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

15. curve fitting - set up $Ax = b$, and solve by substitution

    (a) find a quadratic $f(x) = a + bx + cx^2$ with $f(0) = 7$, $f(1) = 6$, $f(3) = -14$
    (b) find a cubic $f(x) = a + bx + cx^2 + dx^3$ with $f(1) = 10$, $f(-1) = -2$, $f(2) = 17$, $f(3) = 26$
    (c) find a circle $x^2 + y^2 + ax + by + c = 0$ passing thru $(0, 0)$, $(10, 0)$, and $(0, 4)$
    (d) find a circle $x^2 + y^2 + ax + by + c = 0$ passing thru $(2, 0)$, $(10, 0)$, and $(0, 4)$

16. **Gauss elimination** (German 17th c.) **algorithm** (Persian 9th c.); systematic method to solve linear systems.

17. write $Ax = b$ as **augmented matrix** $\hat{A} = [A|b]$, and use **elementary row operations (ERO)**

    - work your way from NW to SE getting a leading 1 (**pivot**) in each row
    - get zero's below each pivot
    - work your way back up from SE to NW, getting zeros above each pivot

    If A is square and a unique solution exists, the result will be 1's on the main (Gordon) diagonal, and the augmented matrix will reveal the solution as $[I|x]$.

18. ERO's generate **equivalent** (symbol $\sim$) linear systems

    - swap rows, $R_i \leftrightarrow R_j$
    - scale a row, αR_i
    - subtract rows, $R_i \leftarrow R_i - \alpha R_j$, or $R_i \mathrel{-}= \alpha R_j$

    illustrate with apple,banana,clementine prices

19. Some easy cases:

    - diagonal - just need to scale the rows
    - triangular - can easily describe ERO's, but in practice just use **back-substitution**

$$\left[\begin{array}{ccc|c} 1 & 5 & 7 & * \\ 0 & 2 & 6 & * \\ 0 & 0 & 4 & * \end{array}\right]$$

ERO's: $\frac{1}{2}R_2, \frac{1}{4}R_3, R_2 \mathrel{-}= 3R_3, R_1 \mathrel{-}= 7R_3, R_1 \mathrel{-}= 5R_2$

20. Octave `rref(Ahat)`

21. a **square** matrix has $m = n$ (same number of rows as columns)

22. a matrix is **symmetric** if $A^T = A$; this implies A is square

23. So far, we're only considering square A. What if `rref(Ahat)` has a row of zeros at the bottom? Then there is a **degree of freedom**; the variable without a pivot 1 in its column can be set arbitrarily.

24. Find 3 different solutions to each of these:

    - $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 8 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$
    - $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
    - $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
    - $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

25. a **diagonal** matrix could be written e.g. $D = \text{diag}(2, 5, -1)$, and has $d_{ij} = 0$ unless $i = j$

26. **identity** matrix is square with 1's on the Gordon line and zeros elsewhere. Notice that $Ix = x$.

27. matrix scalar multiplication: αA scales each entry by α

28. matrix addition: if A and B have the same dimensions, then just add the corresponding entries

29. if the dimensions don't match, then scalar multiplication and addition are undefined

30. examples: make up some matrices and compute expressions like

    - $A + 3I$
    - $\frac{1}{2}(A + A^T)$

31. VT 1114 2.1.2 - discuss notation, show that $A = \frac{1}{10}I$

    The (4×4) matrix $A = (a_{ij})$ is added to itself 9 times (total of 10 A's) to get matrix B.

    $$A + A + A + \cdots + A + A = B$$

    where $B = (b_{ij})$,

    $$b_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

    Which one of the following represents matrix A?

32. matrix multiplication: let $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$, define $C = AB$

    - do not just multiply corresponding entries
    - the **inner dimensions** match (p), and the size of the product is $m \times n$
    - $C = [Ab_1, Ab_2, \cdots, Ab_p]$, i.e. each column is a linear combination of A's columns
    - $c_{ij} = a_{i:} \cdot b_j$, i.e. the dot product of the ith row of A with the jth column of B
    - generally **not commutative** $AB \neq BA$
    - is **associative** $ABC = (AB)C = A(BC)$, e.g. $(3 \times 5)(5 \times 2)(2 \times 8)$ **produces** a 3×8
    - $(A + B)^2 = (A + B)(A + B) \neq A^2 + 2AB + B^2$

33. example: a_{ij} is the amount of nutrient (calories,fiber,vitC,sugar,potassium,etc) i in fruit j, and b_{jk} is the number of fruit j in smoothie k; then $C = AB$ computes $c_{ik} = \sum a_{ij}b_{jk}$ the amount of nutrient i in smoothie k

34. you might not need to compute the full matrix
    e.g. to find ABx, compute $A(Bx)$ as two matrix-vector multiplies

35. important cases

    - both A^TA and AA^T are symmetric, but generally not equal
    - **pre-multiplying** by a diagonal matrix scales rows
    - **post-multiplying** by a diagonal matrix scales columns
    - if $x, y \in \mathbb{R}^n$ are column vectors, then

      xy is undefined because $(n \times 1)(1 \times n)$ don't conform
      $x^T y = x \cdot y$ because $(1 \times n)(n \times 1)$ yields a 1×1 (**inner product**)
      xy^T is a $n \times n$ matrix w. each col a multiple of x and each row a mult. of y (**outer product**)

36. On a desert island, three entities (agriculture, government, security) spend a fixed supply of gold coins according to this matrix:

$$A = \begin{bmatrix} .15 & .50 & .35 \\ .40 & 0 & .40 \\ .45 & .50 & .25 \end{bmatrix}$$

    i.e. agriculture hoards 15%, spends 40% in taxes, and 45% on security.

    - suppose the initial distribution is $x^{(0)} = [0; 0; 100]$ (security found the treasure chest).
    - after one period $x^{(0)} = Ax^{(0)} = [35; 40; 25]$
    - after two period $x^{(1)} = A(Ax^{(0)}) = [34; 24; 42]$
    - after a long time $x^{\infty} = A(A(A(A \cdots (Ax^{(0)})))) = [32.74; 28.57; 38.69]$
      this is the **limiting**, **steady state**, **equilibrium** distribution
    - Notice that at equilibrium, $Ax = x$, or $(A - I)x = 0$. Let $B = A - I$.

$$B = \begin{bmatrix} -.85 & .50 & .35 \\ .40 & -1 & .40 \\ .45 & .50 & -.75 \end{bmatrix}$$

    - Letting $\hat{B}$ be the augmented matrix for $Bx = 0$, show that (multiply through by 20 first)

$$rref(\hat{B}) = \left[\begin{array}{ccc|c} 1 & 0 & -11/13 & 0 \\ 0 & 1 & -48/65 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

    - Since there is a row of zeros at the bottom, we have a **free variable**. There is no pivot in column 3, so x_3 could be anything. Let $x_3 = 65$ and use back-substitution to get $x = [55; 48; 65]$.
    - We've got the ratios correct, but $1 \cdot x = 168$ coins. So multiply by $100/168$ to get $[32.74; 28.57; 38.69]$.
    - Another possibility is to append another row to B.

$$rref\left(\left[\begin{array}{ccc|c} -.85 & .50 & .35 & 0 \\ .40 & -1 & .40 & 0 \\ .45 & .50 & -.75 & 0 \\ 1 & 1 & 1 & 100 \end{array}\right]\right)$$

37. mid 90's, instead of islanders trading coins, websites trading viewers via links. Eureka, it's Google!

38. **canonical basis vectors** e.g. $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

39. **linear transformations** (function, map, operator)
    - **linear** means that $L(\alpha x + w) = \alpha L(x) + L(w)$
    - $y = L(x)$, can be represented as a matrix multiplication $y = Ax$
    - entirely determined by what it does to the basis, i.e. the columns of A since $a_j = Ae_j$
    - geometric transformations: scale, shear, reflect, rotate
    - square maps to parallelogram, circle maps to ellipse
    - on paper - see what happens to the unit square
    - experiment with $\mathbb{R}^2$ online - try to anticipate the matrix that accomplishes the transformation
    - rotation by an angle θ, $L(e_1) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $L(e_2) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$,
      so $A = \begin{bmatrix}c & -s\\ s & c\end{bmatrix}$ is a **rotation matrix**, and note that each column is a unit vector

40. Visualize: if $L(1,0) = (5,2)$ and $L(0,1) = (3,4)$, then find $L(7,4) = \begin{bmatrix}5 & 3\\ 2 & 4\end{bmatrix}\begin{bmatrix}7\\4\end{bmatrix}$.

41. watch "essence of linear algebra" by 3blue1brown

42. composition of linear transformations is done by matrix multiplication
    - like composition of functions $f(g(x))$
    - apply in sequence on the left, e.g. first A, then B
      $B(Ax) = (BA)x$, so the matrix BA represents the net effect
    - associativity works, e.g. $(AB)(Cx) = A(BC)x$ etc. would be C first, then B, then A
    - stretch $A = \begin{bmatrix}3 & 0\\ 0 & 1\end{bmatrix}$, then rotate $B = \begin{bmatrix}c & -s\\ s & c\end{bmatrix}$, or vice-versa. Show that $AB \neq BA$ geometrically and algebraically.
    - but successive rotations do commute

43. things a matrix multiplication accomplishes
    - combining information
    - moving a system from one time to another
    - geometric transformations

44. to **invert** means to undo (ctrl-z) by doing the opposite
    (restore/recover original, backwards in time)
    - recall function notation $f^{-1}(f(x)) = x$
    - we want $A^{-1}Ax = x$ for all x, implying that $A^{-1}A = I$

45. Geometric intuition to invert basic linear transformation matrices.
    - diagonal scaling $D = \begin{bmatrix}a & 0\\ 0 & b\end{bmatrix}$, then $D^{-1} = \begin{bmatrix}1/a & 0\\ 0 & 1/b\end{bmatrix}$
    - rotation $R = \begin{bmatrix}c & -s\\ s & c\end{bmatrix}$, then $R^{-1} = \begin{bmatrix}c & s\\ -s & c\end{bmatrix}$

- shear $S = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then $S^{-1} = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix}$ (think deck of cards)

46. It doesn't matter which order: $D^{-1}D = DD^{-1} = R^{-1}R = RR^{-1} = S^{-1}S = SS^{-1} = I$. All matrices commute with their inverses.

47. To invert a composition, e.g. $T = RS$, undo one at a time (see Sage demo) $(S^{-1}R^{-1})(RS) = I$, so $T^{-1} = S^{-1}R^{-1}$.

48. Observe a pattern for 2×2 inverses? Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The **determinant** of A is $\det(A) = ad - bc$. Unless $\det(A) = 0$, a unique inverse exists.

49. examples

    (a) let $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$; solve $Ax = b$ by pre-multiplying by A^{-1}

    (b) for what x is $A = \begin{bmatrix} x & 3 \\ 8 & x+2 \end{bmatrix}$ not invertible (**singular**) ?

    (c) $A \in \mathbb{R}^{2\times 2}$, $a_{ij} = \frac{i}{i+j+1}$, find A^{-1}

    (d) $A = \begin{bmatrix} .8 & -.6 \\ .6 & .8 \end{bmatrix}$, find a matrix with $M^2 = I$ (rotate half as much)

    (e) create integer matrices with $\det(A) = 1$ and $\det(B) = 1$ (so inverse entries are integers) show that $(AB)^{-1} = B^{-1}A^{-1}$ and notice that $\det(AB) = 1$

    (f) $A = \begin{bmatrix} 7 & 2 \\ 9 & 3 \end{bmatrix}$, find $A^{-4} = (A^{-1})^4$

50. Let $A \in \mathbb{R}^{n\times n}$. An inverse must satisfy $AX = I$, so

$$Ax_1 = e_1, \qquad Ax_2 = e_2, \qquad \cdots \qquad Ax_n = e_n$$

Represent as an augmented matrix, and use Gauss elimination to get rref. If there are no zero rows, then $\det(A) \neq 0$, and the inverse is revealed: $[A \mid I] \sim [I \mid A^{-1}]$.

51. Model spread of coronavirus via matrix multiplication. Each person contagious for two weeks, and each contagious person infects one other person each week. $1, 1, 2, 3, 5, \cdots$ (see Sage demo, golden ratio)

52. You can **factor** any (invertible) 2×2 matrix as $A = RSD$. (see Sage demo)

53. You can factor a (invertible) 2×2 matrix as $A = D_1S_1D_2S_2$. (see Sage demo) Apply matrices on the left representing row operations:

$$\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2/11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Write A as the product of those inverted components.

$$\begin{bmatrix} 4 & 2 \\ 3 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11/2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

(note: you could wait until end to do all the scaling; and we are not permuting rows at this point)

54. **QR factorization**: write $A = QR$ where $Q^TQ = I$ (so $Q^{-1} = Q^T$) and R is upper triangular. Here's how you can do it in the 2×2 case:

   (a) Let $\theta = \tan^{-1}(a_{21}/a_{11})$.

   (b) Let $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

   (c) Let $R = Q^T A$.

55. **LU factorization**: write $A = LU$ where L and U are lower and upper triangular respectively. Here's how you can do it in the 2×2 case:

   (a) Let $L = \begin{bmatrix} 1 & 0 \\ a_{21}/a_{11} & 1 \end{bmatrix}$. (shear based on a row operation)

   (b) Let $U = L^{-1}A$.

56. Let v be a vector, and let $Px = cv$ be the **orthogonal projection** of x onto v (where c is a constant). Sketch a picture with $x = cv + w$ where $w^Tv = 0$. Then

$$\begin{aligned} v^Tx &= v^T(cv + w) \\ v^Tx &= c(v^Tv) + v^Tw \\ c &= \frac{v^Tx}{v^Tv} \end{aligned}$$

   This c is called the **scalar component** of x on v. Then rearranging we get the projection itself:

$$Px = cv = vc = v\frac{v^Tx}{v^Tv} = \frac{vv^T}{v^Tv}x$$

   So the **projection matrix** is $P = \frac{vv^T}{v^Tv}$.

57. Things to note about projections:

   - Information is lost. A higher dimensional object becomes a shadow of itself.
   - $P^2 = P$. Project again has no effect.
   - $P^T = P$. This is true for orthogonal projections.

58. Let θ be the angle between x and v.

$$\cos\theta = \frac{\|Px\|}{\|x\|} = \sqrt{\frac{(Px)^T(Px)}{x^Tx}} = \sqrt{\frac{x^TP^TPx}{x^Tx}} = \sqrt{\frac{x^TPx}{x^Tx}} = \sqrt{\frac{x^Tvv^Tx}{v^Tvx^Tx}} = \frac{v^Tx}{\|v\|\|x\|}$$

   Use this formula to find the angle between vectors.

59. We can over or under project by using the **flattening operator**

$$F(\alpha) = (1 - \alpha)P + \alpha I$$

   - $F(0) = P$ (complete pancake)
   - $F(1) = I$ (does nothing)
   - $F(-1) = 2P - I$ is a **reflection** across v
   - $F(.2) = .8P + .2I$ (squished around v)
   - $F(2) = 2I - P$ (stretched around v)

60. Let's find a matrix that reflects across the line $y = mx$. Let $v = \begin{bmatrix} 1 \\ m \end{bmatrix}$

$$P = \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

$$F(-1) = 2P - I = \frac{1}{1+m^2} \begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

61. Examples - March 16

(a) Find the angle between $v = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ and $x = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$.

Answer: use the formula $\theta = \cos^{-1}\left(\frac{v^T x}{\|v\|\|x\|}\right) = \cos^{-1}\left(\frac{64}{\sqrt{53}\sqrt{80}}\right) = 10.62°$

(b) Find the slope of the line $y = mx$ that intersects $y = 7 - 2x$ at a $20°$ angle.

Answer: Form two vectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ m \end{bmatrix}$ that lie parallel to the given lines. Then we want $\cos(20) = \frac{1-2m}{\sqrt{5}\sqrt{1+m^2}}$

$$(\cos(20)\sqrt{5})^2 = \frac{(1-2m)^2}{1+m^2}$$

$$4.4151(1+m^2) = (1-2m)^2$$

solve this quadratic by hand, or with WolframAlpha to get: $m \approx -.9468, -8.6894$

(c) Find the orthogonal projection matrix onto the line $y = 2x$. Project $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ onto that line.

Answer: Use the vector $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and compute $P = \frac{vv^T}{v^T v} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $Px = \begin{bmatrix} 11/5 \\ 22/5 \end{bmatrix}$

(d) Find matrix that orthogonally reflects across the line $y = 2x$. Reflect $\vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ across that line.

Answer: $R = 2P - I = \frac{2}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$. Then $Rx = \begin{bmatrix} -3/5 \\ 29/5 \end{bmatrix}$

(e) Suppose $vv^T = \begin{bmatrix} 49 & -21 \\ -21 & 9 \end{bmatrix}$.

i. Find v if $v_1 > 0$.

Answer: $v_1^2 = 49$ and $v_2^2 = 9$ and $v_1 v_2 = -21$, so $v = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$

ii. Find the $\perp$ projection matrix P onto v.

Answer: $P = \frac{1}{58}\begin{bmatrix} 49 & -21 \\ -21 & 9 \end{bmatrix}$

iii. Find the $\perp$ reflection matrix R across v.

Answer: $R = 2P - I = \frac{1}{29}\begin{bmatrix} 20 & -21 \\ -21 & -20 \end{bmatrix}$

(f) Let $v = \begin{bmatrix} .6 \\ .8 \end{bmatrix}$, which is a unit vector. Then $P = vv^T = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$. Let $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute and sketch (see `https://massey.limfinity.com/208/sage.htm`)

i. $F(1) = 0P + 1I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $F(1)x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. (does nothing)

ii. $F(0) = 1P + 0I = \begin{bmatrix} .36 & .48 \\ .48 & .64 \end{bmatrix}$, so $F(0)x = \begin{bmatrix} .48 \\ .64 \end{bmatrix}$ (project onto v)

iii. $F(-1) = 2P - I = \begin{bmatrix} -.28 & .96 \\ .96 & .28 \end{bmatrix}$, so $F(-1)x = \begin{bmatrix} .96 \\ .28 \end{bmatrix}$ (reflect across v)

iv. $F(.3) = .7P + .3I = \begin{bmatrix} .552 & .336 \\ .336 & .748 \end{bmatrix}$, so $F(.3)x = \begin{bmatrix} .336 \\ .748 \end{bmatrix}$ (flatten around v)

v. $F(2) = -P + 2I = \begin{bmatrix} 1.64 & -.48 \\ -.48 & 1.36 \end{bmatrix}$, so $F(2)x = \begin{bmatrix} -.48 \\ 1.36 \end{bmatrix}$ (stretch around v)

62. Wed, Mar 18 - Example 1
Let $A \in \mathbb{R}^{4\times 2}$ with $a_{ij} = i - 2j$.

(a) Write A.

$$A = \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$$

(b) Find $B = A^T A$.

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -3 & -2 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix}$$

(c) Find $(A^T A)^{-1}$. Note that $\det(B) = (6)(14) - (2)(2) = 80$

$$\frac{1}{80} \begin{bmatrix} 14 & -2 \\ -2 & 6 \end{bmatrix}$$

(d) Find a LU factorization of B. Do one row operation to get U.

$$\begin{bmatrix} 1 & 0 \\ -1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 0 & 40/3 \end{bmatrix}$$

Putting the shear on the other side, we get $B = LU$ as:

$$\begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 0 & 40/3 \end{bmatrix}$$

(e) Find a QR factorization of B. Let $\theta = \tan^{-1} 2/6$, and let $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} .949 & -.316 \\ .316 & .949 \end{bmatrix}$.
Then let

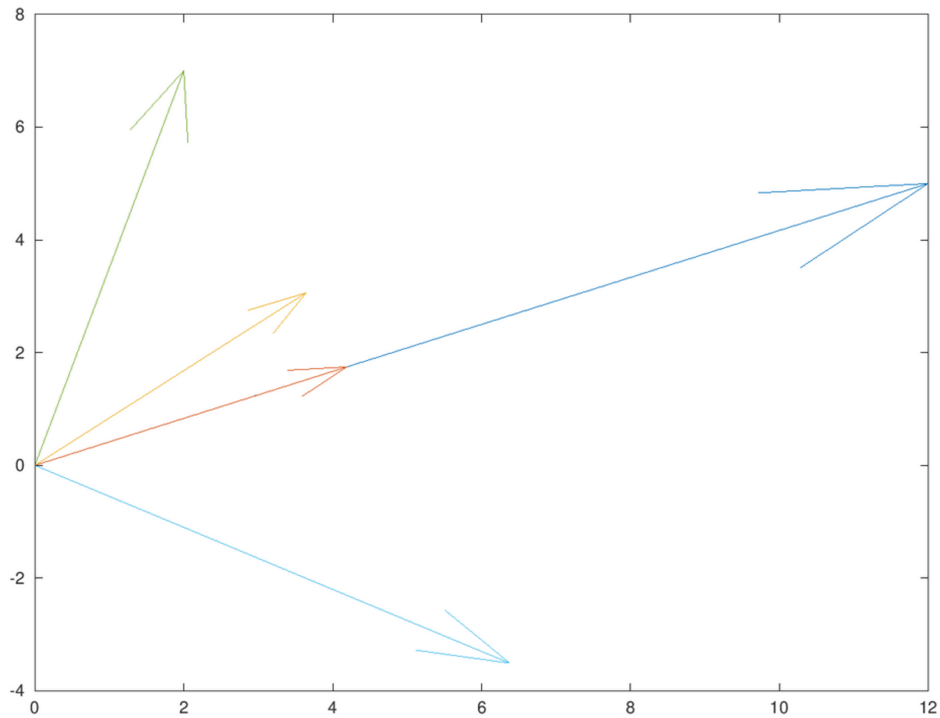
$$R = Q^T B = \begin{bmatrix} .949 & -.316 \\ .316 & .949 \end{bmatrix} \begin{bmatrix} 6 & 2 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 6.325 & 6.325 \\ 0 & 12.65 \end{bmatrix}$$

You can check that now $QR = B$.

63. Wed, Mar 18 - Example 2

Let $v = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$ (long dark blue), and $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ (green).

You may try this yourself at `https://massey.limfinity.com/208/sage.htm`.



(a) Find the angle between v and x.

$$\cos\theta = \frac{v^T x}{\|v\|\|x\|} = \frac{59}{\sqrt{169}\sqrt{53}} = .62341$$

$$\theta = \cos^{-1}(.62341) = .89771 = 51.4^\circ$$

(b) Find the matrix P that projects onto the line $y = \frac{5}{12}x$. Note that this line is just the extension (span) of the vector v.

$$P = \frac{vv^T}{v^T v} = \frac{1}{12^2 + 5^2}\begin{bmatrix} 12 \\ 5 \end{bmatrix}[12\ 5] = \frac{1}{169}\begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix}$$

(c) Project the vector $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ onto v. (orange)

$$Px = \frac{1}{169}\begin{bmatrix} 144 & 60 \\ 60 & 60 \end{bmatrix}\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 708/169 \\ 295/169 \end{bmatrix}$$

(d) Find the matrix R that reflects across the line $y = \frac{5}{12}x$.

$$2P - I = \frac{2}{169}\begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{169}\begin{bmatrix} 119 & 120 \\ 120 & -119 \end{bmatrix}$$

(e) Reflect the vector $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ across v. (light blue)

$$Rx = (2P - I)x = 2Px - x = 2 \begin{bmatrix} 708/169 \\ 295/169 \end{bmatrix} - \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \frac{1}{169} \begin{bmatrix} 1078 \\ -593 \end{bmatrix}$$

(f) Form the "flattening" matrix $F(.25) = .75P + .25I$, which essentially does 75% of the projection, leaving vectors 25% as far from v as they started.

$$F(.75) = \frac{3}{4} \frac{1}{169} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{169} \begin{bmatrix} 601 & 45 \\ 45 & 61 \end{bmatrix}$$

(g) Flatten the vector $x = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$ around v. (yellow)

$$F(.75)x = (.75P + .25I)x = .75Px + .25x = .75 \begin{bmatrix} 708/169 \\ 295/169 \end{bmatrix} + .25 \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 1231/338 \\ 517/169 \end{bmatrix}$$

Determinants

64. Consider a square $n \times n$ matrix A, representing a linear transformation. The unit n-dimensional cube is transformed into a parallelpiped with edges given by the columns of A. The **determinant** of A, written $\det(A)$ (or sometimes $|A|$) gives the signed volume of that parallelpiped.

65. Octave `det(A)`.

66. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$, then $\det(A) = ad - bc$.

    - This is the area of the parallelogram defined by vectors $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$.
    - The area of the triangle $(0,0)$, (a,c), (b,d) is $\frac{1}{2}\det(A)$.

67. A negative determinant tells you that the transformation changed the **orientation** of space (has turned the cube "inside out").

68. Here are some useful properties of determinants (see Strang video).

    (1) $\det(I) = 1$

    (2) If you swap two rows of A, the determinant switches sign (multiply by -1).

    (3) The det function is linear in any particular row.
      - multiply a row by c, and the determinant is multipled by c
      - $\det(A(i,:) \triangleq u + v) = \det(A(i,:) \triangleq u) + \det(A(i,:) \triangleq v)$

    (4) if two rows are equal, then $\det(A) = 0$

    (5) a "shear" (adding a multiple of one row to another) does not change the determinant

    (6) if there is a row of zeros, then $\det(A) = 0$

    (7) the determinant of a triangular matrix is the product of the diagonal (Gordon line) entries

    (8) $\det(A) = 0$ if and only A is **singular** (not invertible)

    (9) $\det(AB) = \det(A)\det(B)$

    (10) $\det(A^T) = \det(A)$

    (11) $\det(A^{-1}) = \frac{1}{\det(A)}$

    (12) $\det(cA) = c^n \det(A)$

69. Here are two ways to compute a determinant:

    - Use row operations to reduce A to upper-triangular. The determinant is the product of the diagonals, times -1 each time you had to do a row swap.

    $$\det \begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ -2 & 4 & 8 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3.5 & 1.5 \\ 0 & 7 & 9 \end{bmatrix} = \det \begin{bmatrix} 2 & 3 & 1 \\ 0 & 3.5 & 1.5 \\ 0 & 0 & 6 \end{bmatrix} = 42$$

    The row operations were: $R_2 \mathrel{-}= \frac{1}{2}R_1, R_3 \mathrel{+}= R_1, R_3 \mathrel{-}= 2R_2$

    - Expand across any row or column using **cofactors**, e.g. across row i:

    $$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij}$$

where the cofactor C_{ij} is $(-1)^{i+j}$ times the determinant of the matrix with row i and column j removed (the ij **minor**).

$$\det\begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ -2 & 4 & 8 \end{bmatrix} = -3\det\begin{bmatrix} 1 & 2 \\ -2 & 8 \end{bmatrix} + 5\det\begin{bmatrix} 2 & 1 \\ -2 & 8 \end{bmatrix} - 4\det\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = -(3)(12)+(5)(18)-(4)(3) = 42$$

Note we expanded down the 2nd column, starting with a negative since $(-1)^{(1+2)} = -1$.

70. Find the determinant of this matrix both ways: $A = \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix}$

- Document what row operations I am doing here:

$$\det\begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 3 & 2 & 1 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & -5 & 0 \\ 0 & 0 & 5 & 4 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 5 & 4 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & -5 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

  The product of the diagonals is -30, but since we had to do a swap, $\det(A) = -(-30) = 30$.

- The cofactor method is not too bad when there are a lot of zeros. Let's go across the last row:

$$\begin{aligned}
\det(A) &= -5\det\begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} + 4\det\begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} \\
&= -5(-1\det\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}) + 4(-2\det\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}) \\
&= -5(-1(-2)) + 4(-2(-5)) \\
&= -10 + 40 \\
&= 30
\end{aligned}$$

71. The **adjoint** of a square matrix is the transpose of the matrix of cofactors. e.g.

$$\text{adj}\left(\begin{bmatrix} 2 & 3 & 1 \\ 1 & 5 & 2 \\ -2 & 4 & 8 \end{bmatrix}\right) = \begin{bmatrix} 32 & -12 & 14 \\ -20 & 18 & -14 \\ 1 & -3 & 7 \end{bmatrix}^T = \begin{bmatrix} 32 & -20 & 1 \\ -12 & 18 & -3 \\ 14 & -14 & 7 \end{bmatrix}$$

Consider where the -12 comes from: cross out the 1st row and 2nd column to get

$$(-1)^{1+2} \det \begin{bmatrix} 1 & 2 \\ -2 & 8 \end{bmatrix} = -12$$

72. If $\det(A) \neq 0$, then A is invertible, and

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

So in the previous example,

$$A^{-1} = \frac{1}{42} \begin{bmatrix} 32 & -20 & 1 \\ -12 & 18 & -3 \\ 14 & -14 & 7 \end{bmatrix}$$

We won't worry about **Cramer's Rule** and how this can be used to solve linear systems (it's not efficient).

73. (see videos) Consider a parallelpiped (slanted box) with edges $a_1, a_2, \cdots a_n$ emanating from the origin. Recognize that the volume has essentially the same properties as the deteriminant (scaling and shearing in particular). A negative determinant means the box is "inside-out" compared with the standard basis.

$$\text{volume parallelpiped} = |\det A|$$

As a special case, in $\mathbb{R}^2$, this is the area of a parallelogram, so

$$\text{area triangle} = \frac{1}{2} |\det A|$$

- area of triangle $(0,0)$, $(4,9)$, $(8,3)$ is $\frac{1}{2}|\det \begin{bmatrix} 4 & 8 \\ 9 & 3 \end{bmatrix}| = 30$
- volume of parallelpiped represented by columns of $A = \begin{bmatrix} 7 & 2 & 1 \\ 0 & 9 & 3 \\ 2 & -1 & 6 \end{bmatrix}$ is $|\det A| = 393$
- volume of a pyramid with those three columns as lateral edges is $\frac{1}{6}|\det A| = 65.5$

74. Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be coordinates of a triangle's corners. We could shift the coordinates so that one point is at the origin, and then algebraically check that:

$$\text{area triangle} = \frac{1}{2} \det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} = \frac{1}{2} \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{bmatrix}$$

Visualize this geometrically: let A be this last 3×3 matrix, which is the triangle's corners with $z = 1$ coordinates appended, lifting the triangle to create a pyramid with apex at the origin and triangle as the base. The volume of a pyramid is $\frac{1}{3}$(height)(area triangle) $= \frac{1}{6}|\det(A)|$. Since we set the height equal 1, area triangle $= \frac{1}{2}|\det(A)|$.

e.g. consider $(2, -5), (6, 4), (10, -2)$, then $\frac{1}{2} \det \begin{bmatrix} 2 & 6 & 10 \\ -5 & 4 & -2 \\ 1 & 1 & 1 \end{bmatrix} = 30$

75. When working in $\mathbb{R}^3$, there is a mysterious object called the **cross product**, which you may learn more about in physics or calculus class. Let $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then we define

$$u \times v = \begin{bmatrix} u_2v_3 - v_2u_3 \\ -(u_1v_3 - v_1u_3) \\ u_1v_2 - v_1u_2 \end{bmatrix}$$

This can be remembered as following the pattern of a determinant (see links). Here are some defining properties:

- $u \times v$ is $\perp$ both u and v (check that the dot product is zero)
- $\|u \times v\| = \|u\|\|v\| \sin\theta$, where θ is the angle between u and v
- $u \times v$ obeys the **right-hand-rule** ($\det[u, v, u \times v] \geq 0$)

The area of a triangle with sides u and v (emanating from a corner) is $\frac{1}{2}\|u \times v\|$. e.g. recall the previous example triangle, but append a zero for the third coordinate: $(2, -5, 0), (6, 4, 0), (10, -2, 0)$, then $\frac{1}{2}\| \begin{bmatrix} 4 \\ 9 \\ 0 \end{bmatrix} \times \begin{bmatrix} 8 \\ 3 \\ 0 \end{bmatrix} \| = \frac{1}{2}\| \begin{bmatrix} 0 \\ 0 \\ 12 - 72 \end{bmatrix} \| = 30$

76. Examples:

- Let $u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$. Find a vector orthogonal to both u and v.

  Answer: $u \times v = \begin{bmatrix} 2 - 6 \\ -(1 - 12) \\ 2 - 8 \end{bmatrix} = \begin{bmatrix} -4 \\ 11 \\ -6 \end{bmatrix}$

- Let $P(4, 2, 7)$, $Q(5, 8, 12)$, and $R(9, 4, 10)$ be three points in $\mathbb{R}^3$. Find the area of $\triangle PQR$.

  Answer: let $u = \vec{PQ} = Q - P = \begin{bmatrix} 1 \\ 6 \\ 5 \end{bmatrix}$ and $v = \vec{PR} = R - P = \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$.

  The area is $\frac{1}{2}\|u \times v\| = \frac{1}{2}\| \begin{bmatrix} 18 - 10 \\ -(3 - 25) \\ 2 - 30 \end{bmatrix} \| = \frac{1}{2}\|[8; 22; -28]\| = \frac{1}{2}\sqrt{8^2 + 22^2 + 28^2} = \frac{1}{2}\sqrt{1332}$.

77. Let's review solving a linear system in which there is a row of zeros (usually after doing row operations). In this class, we'll focus on square matrices with one row of zeros. This indicates that there is one free variable. We are often interested in a **non-trivial** solution ($x \neq 0$) to a **homogeneous system** $Ax = 0$. In many cases, you can find a solution **by inspection**; i.e. just "see" what would work.

   - $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
     We can let x_3 be anything we want, say $x_3 = 1$. The 2nd row says: $4x_2 + 5(1) = 0$, so $x_2 = -5/4$. Then $x_1 + 2(-5/4) + 3(1) = 0$ says $x_1 = -1/2$. So $x = \begin{bmatrix} -1/2 \\ -5/4 \\ 1 \end{bmatrix}$ is a solution. But because $Ax = 0$, any multiple of x is also a solution. Multiply by 4 to get rid of the fractions: $\begin{bmatrix} -2 \\ -5 \\ 4 \end{bmatrix}$.
   - $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$
     The 2nd row forces $x_2 = 0$. Then x_3 is free, so say $x_3 = 1$. Then $x_1 + 2(0) + 3(1) = 0$ says $x_1 = -3$. So a solution is $x = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Can you "see" this by inspection ?
   - $\left[\begin{array}{ccc|c} 2 & 8 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. By inspection, $x = \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$ works.

78. Let A be a square matrix. Then x is an **eigenvector**, with **eigenvalue** λ, if

$$Ax = \lambda x$$

   In other words, A scales, but doesn't rotate x. So Ax is parallel to x.

   - We may refer to (λ, x) as an **eigenpair**.
   - Generally λ could be a complex number (important for understanding rotational behavior), but in this class we will consider only real valued λ.
   - An eigenvector must be **non-trivial**, i.e. $x \neq 0$
   - Suppose $Ax = \lambda x$, then $A(cx) = c(Ax) = c(\lambda x) = \lambda(cx)$. So any non-zero multiple of an eigenvector is also an eigenvector for the same eigenvalue. Therefore any non-trivial eigenvector is fine. e.g. if $x = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$ is an eigenvector, then so is $\begin{bmatrix} -4 \\ 18 \end{bmatrix}$.
   - If A is $n \times n$, then A could have up to n distinct eigenvalues. (In more advanced classes, we allow for complex numbers and repeated eigenvalues, so that we can say there are n eigenvalues.)
   - Eigenvalues are generally not easy to find. We'll stick to the "easy" cases.
   - The eigenpairs **characterize** a natural basis for understanding the long-term behavior of a system that evolves via multiplication by A. (very important in engineering, physics, networks - see video)

79. Let A be a square matrix. This is a partial list of equivalent statements (one implies them all) known as the **invertible matrix theorem**.

   - A is singular (not invertible)
   - $\det(A) = 0$

- $rref(A)$ has a row of zeros
- $Ax = 0$ has a non-trivial solution
- 0 is an eigenvalue of A

80. Suppose (λ, x) is an eigenpair. Then $Ax = \lambda x$. Move everything to one side:

$$(A - \lambda I)x = 0$$

So one way to find the eigenvalues is to set $\det(A - \lambda I) = 0$. Then solve homogeneous linear systems to find the corresponding eigenvectors. For example, find the eigenpairs of $A = \begin{bmatrix} 5 & 7 \\ 9 & 3 \end{bmatrix}$.

Answer: $0 = \det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & 7 \\ 9 & 3-\lambda \end{bmatrix} = (5-\lambda)(3-\lambda) - 63 = \lambda^2 - 8\lambda - 48 = (\lambda - 12)(\lambda + 4)$, so the two eigenvalues are $\lambda = 12, -4$. NOTE: if you can't factor the **characteristic polynomial**, use the quadratic formula.
If $\lambda = 12$, then solve $\begin{bmatrix} -7 & 7 \\ 9 & -9 \end{bmatrix} x = 0$ to get $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
If $\lambda = -4$, then solve $\begin{bmatrix} 9 & 7 \\ 9 & 7 \end{bmatrix} x = 0$ to get $x = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$.

81. If A is symmetric, then it's a fact that the eigenvalues are real numbers, and the eigenvectors are orthogonal. Find the eigenpairs of $A = \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$.

Answer: $\det \begin{bmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix} = 0$ says $(5-\lambda)^2 - 4 = 0$, so $5 - \lambda = \pm 2$, and $\lambda = 3, 7$.
If $\lambda = 3$ then solve $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} x = 0$ to get $x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
If $\lambda = 7$ then solve $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} x = 0$ to get $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

82. Let $v = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$, then $P = \frac{vv^T}{v^Tv} = \frac{1}{169}\begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix}$ is a projection matrix.

- Since $w = \begin{bmatrix} -5 \\ 12 \end{bmatrix}$ is orthogonal to v, we know geometrically that $Pw = 0$. Therefore zero is an eigenvalue of P.
- Also, since $Pv = v$, we know $\lambda = 1$ is an eigenvalue.

Therefore the eigenpairs are: $\{(0, w), (1, v)\}$.

83. Consider a rotation matrix $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

- If $\theta = 0$ (or a multiple of 2π), then $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so the only eigenvalue is 1, and everything is an eigenvector: $Ax = 1x$
- If $\theta = \pi$ (plus some multiple of 2π), then $Q = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, so the only eigenvalue is -1, and everything is an eigenvector: $Ax = -1x$
- Otherwise, there is no real eigenpair, because every vector gets rotated off it's original line. (see videos for the 90° case.)

84. The **trace** of a matrix is the sum of the diagonal entries (Gordon line). An amazing fact is that:

- The sum of the eigenvalues equals the trace.
- The product of the eigenvalues equals the determinant.

For example, suppose $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix}$ and you are told $\lambda_1 = 2$, then what is the other eigenvalue λ_2 ?

Answer: $2 + \lambda_2 = 5 + 4 = 9$ and $2\lambda_2 = 14$, and either equation tells you $\lambda_2 = 7$

85. Generally the eigenvalues of a triangular matrix are the diagonal (Gordon line) entries.

e.g. Find the eigenpairs of the triangular matrix $A = \begin{bmatrix} 2 & 9 \\ 0 & 5 \end{bmatrix}$.

Answer: $(2 - \lambda)(5 - \lambda) = 0$, so $\lambda = 2, 5$

If $\lambda = 2$, then solve $\begin{bmatrix} 0 & 9 \\ 0 & 3 \end{bmatrix} x = 0$, so $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

If $\lambda = 5$, then solve $\begin{bmatrix} -3 & 9 \\ 0 & 0 \end{bmatrix} x = 0$, so $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

86. For anything bigger than 2×2, finding eigenpairs by hand is a nightmare. See the video for a worked out 3×3 example. Setting $\det A = 0$ results in a **characteristic polynomial** in λ that you must find the roots of. Only in special cases is this even feasible. See the Octave demo for use of `[V,L] = eig(A)`

87. As is typical for many computational problems, even if it's difficult to **find** a solution, it is relatively **easy to check** that a solution is correct. For example, verify that $x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a eigenvector of $A = \begin{bmatrix} 8 & -5 & 3 \\ -2 & -1 & 6 \\ 5 & -7 & 10 \end{bmatrix}$, and find the associated eigenvalue.

Answer: $Ax = \begin{bmatrix} 7 \\ 14 \\ 21 \end{bmatrix}$, so the eigenvalue is $\lambda = 7$.

88. Find the eigenpairs of $A = \begin{bmatrix} 5 & 7 & 0 \\ 9 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Answer: Fortunately, this matrix is **block diagonal**. Refer back to an earlier example with $\begin{bmatrix} 5 & 7 \\ 9 & 3 \end{bmatrix}$.

Check that the eigenpairs are: $(12, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix})$, $(-4, \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix})$, and $(2, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$.

89. Find the eigenpairs of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Answer: Since A is triangular, the eigenvalues are $1, 4, 6$.

If $\lambda = 1$, solve $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

If $\lambda = 4$, solve $\begin{bmatrix} -3 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Let $x_2 = 1$, so then $x = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix}$.

If $\lambda = 6$, solve $\begin{bmatrix} -5 & 2 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Let $x_3 = 1$, so then $x = \begin{bmatrix} 8/5 \\ 5/2 \\ 1 \end{bmatrix}$.

90. If a matrix has a full set of independent eigenvectors, we can put them in a matrix V, and the corresponding eigenvalues in a diagonal matrix Λ. Then:

$$AV = V\Lambda$$

We could also write this as the **diagonalization**, showing that A is **similar** to the diagonal matrix Λ.

$$A = V\Lambda V^{-1}$$

Here are the previous examples following that pattern.

- $\begin{bmatrix} 5 & 7 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 1 & 9 \end{bmatrix} \begin{bmatrix} 12 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 1 & 9 \end{bmatrix}^{-1}$
- $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1}$
- $\begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} = \begin{bmatrix} -5 & 12 \\ 12 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 12 \\ 12 & 5 \end{bmatrix}^{-1}$
- $\begin{bmatrix} 2 & 9 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-1}$
- $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{bmatrix}^{-1}$

91. It's not always obvious what's true about eigenpairs. Here are some examples:

- Since $\det(A^T - \lambda I) = \det(A - \lambda I)$, a matrix has the same eigenvalues as its transpose. BUT, the eigenvectors are probably different.
- The eigenvalues of $A + cI$ are c plus the eigenvalues of A.
- The eigenvalues of $A + B$ do not generally equal the eigenvalues of A plus the eigenvalues of B.
- If you double a matrix, the eigenvalues double, and the eigenvectors stay the same.

92. Consider the wealth transfer $x^{(k+1)} = Ax^{(k)}$, where $A = \begin{bmatrix} .1 & .2 & .5 \\ .6 & .2 & .2 \\ .3 & .6 & .3 \end{bmatrix}$

- Recall that A moves us forward in time, e.g. $x^{(3)} = A^3 x^{(0)}$.
- Since each column adds up to 1, we see that $A^T \vec{1} = \vec{1}$. Therefore $\lambda = 1$ is an eigenvalue.
- The corresponding eigenvector is the equilibrium wealth we've discussed earlier in the semester. Suppose for the sake of argument that the total wealth is \$ 100.
- $0 = \det(A - 1I) = \det \begin{bmatrix} .1-\lambda & .2 & .5 \\ .6 & .2-\lambda & .2 \\ .3 & .6 & .3-\lambda \end{bmatrix}$ would be painful to solve by hand.
- Here is Octave code and output:

```
octave:30> A=[.1 .2 .5; .6 .2 .2; .3 .6 .3]
A =

  0.1000000   0.2000000   0.5000000
  0.6000000   0.2000000   0.2000000
  0.3000000   0.6000000   0.3000000
```

```
octave:31> [V,L]=eig(A)
V =

 0.4969293 + 0.0000000i  -0.3162278 + 0.4472136i  -0.3162278 - 0.4472136i
 0.5421047 + 0.0000000i   0.6324555 + 0.0000000i   0.6324555 - 0.0000000i
 0.6776309 + 0.0000000i  -0.3162278 - 0.4472136i  -0.3162278 + 0.4472136i

L =

Diagonal Matrix

 1.0000000 + 0.0000000i                        0                        0
                      0  -0.2000000 + 0.2828427i                        0
                      0                        0  -0.2000000 - 0.2828427i

octave:32> x=V(:,1)
x =

 0.4969293
 0.5421047
 0.6776309

octave:33> norm(x)
ans =  1
octave:34> sum(x)
ans =  1.716665
octave:35> x = x*100/sum(x)
x =

 28.94737
 31.57895
 39.47368
```

- Notice that the only real eigenvalue is $\lambda = 1$, and the reported eigenvector is $v_1 = \begin{bmatrix} .4969 \\ .5421 \\ .6776 \end{bmatrix}$. This is a unit vector but we can scale it arbitrarily. The eigenvector with total \$100 is $x = \begin{bmatrix} 28.95 \\ 31.58 \\ 39.47 \end{bmatrix}$.

93. If we can diagonalize $A = V\Lambda V^{-1}$, then powers of A are easy.

    - $A^2 = (V\Lambda V^{-1})(V\Lambda V^{-1}) = V\Lambda(V^{-1}V)\Lambda V^{-1} = V\Lambda^2 V^{-1}$
    - By the same procedure, $A^k = V\Lambda^k V^{-1}$.
    - Easy to verify that $A^{-1} = V\Lambda^{-1}V^{-1}$.

    See the Octave demo.

94. If all eigenvalues satisfy $|\lambda| < 1$, then A^k will decay to zero.

    - $A = \begin{bmatrix} 4 & 2.04 \\ -6 & -3 \end{bmatrix}$ has eigenvalues .6 and .4; check in Octave that $A^{50} \approx 0$.

- $A = \begin{bmatrix} 4 & 1.96 \\ -6 & -3 \end{bmatrix}$ has eigenvalues 1.2 and .2; check in Octave that A^{50} is blowing up.

95. Create a matrix A that has these eigenpairs: $\{(1.75, \begin{bmatrix} 5 \\ 9 \end{bmatrix}), (-0.3, \begin{bmatrix} 2 \\ 4 \end{bmatrix})\}$.

Answer: Let $V = \begin{bmatrix} 5 & 2 \\ 9 & 4 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} 1.75 & 0 \\ 0 & -0.3 \end{bmatrix}$. Then

$$A = V\Lambda V^{-1} = \begin{bmatrix} 5 & 2 \\ 9 & 4 \end{bmatrix} \begin{bmatrix} 1.75 & 0 \\ 0 & -0.3 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 4 & -2 \\ -9 & 5 \end{bmatrix} = \begin{bmatrix} 20.2 & -10.25 \\ 36.9 & -18.75 \end{bmatrix}$$