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MATH 208 Test 2, Spring 2020

Directions:

- Do not use any notes, books, the internet, or other sources of information.
- You may use a calculator for arithmetic calculations.
- Use notation conventions as described in class.
- You have 55 minutes. You must work alone; do not communicate with any other person.
- To receive full credit, you must **show all relevant work to completely justify your answer.**
- 105 points possible, graded out of 100 points.

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1. (20 pts) Show the row operations a Gaussian Elimination algorithm would use to put this matrix in RREF.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 & 1 \\ -2 & 0 & 1 & 10 & 6 \\ 0 & 1 & 0 & 7 & 3 \end{array}\right]$$

No fractions should appear.
Answer: $R_3 \mathrel{+}= 2R_1, R_2 \leftrightarrow R_4, \frac{1}{5}R_3, R_4 \mathrel{-}= R_3, R_3 \mathrel{-}= 2R_4, R_2 \mathrel{-}= 7R_4, R_1 \mathrel{-}= 2R_3$

2. (10 pts) Let x be solution to a linear system with this augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

If $1 \cdot x = 30$ (i.e. the sum of the entries of x is 30), then find x.
Hint: find any solution, and then scale it (this works only if homogeneous)

Answer: let the free variable $x_3 = 1$, which implies $x_1 = -3$ and $x_2 = -4$. $x = \frac{30}{-6}\begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 20 \\ -5 \end{bmatrix}$

3. (9 pts) Let $A = \begin{bmatrix} 1 & 2 & 8 \\ 2 & 6 & 4 \\ 2 & 3 & 1 \end{bmatrix}$. Find a matrix D such that each column of AD is a unit vector.
Answer: the length's of A's columns are $3, 7, 9$ respectively, so scale them via $D = \text{diag}(1/3, 1/7, 1/9)$

4. (5 pts) If ABC is a 7×4 matrix, A is square, and C has two rows, then what is the size of B ?
Answer: make sure the inner dimensions match: $(7 \times 7)(7 \times 2)(2 \times ?)$, so B is 7×2

5. (18 pts) Consider these matrices:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix} \qquad x = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

(a) Compute BA

Answer: $\begin{bmatrix} 1 & 4 & 4 \\ 4 & 7 & 8 \\ 4 & 8 & 9 \end{bmatrix}$

(b) Compute $x^T(A - 2I)x$

Answer: $(1 \times 3)(3 \times 3)(3 \times 1)$ gives a 1×1, which turns out to be 188

6. (8 pts) Let x and y be column vectors with outer product:

$$xy^T = \begin{bmatrix} 12 & 8 \\ * & 12 \\ 54 & * \end{bmatrix}$$

(where the $*$ entries are a mystery). If $x_1 = 2$, write the vectors x and y.

Answer: $x = \begin{bmatrix} 2 \\ 3 \\ 9 \end{bmatrix}$ and $y = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

7. (6 pts) This augmented matrix is equivalent (via elementary row operations) to the given RREF.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 1 & 3 & 4 & b_2 \\ 1 & 1 & 1 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Find the vector b.

Answer: multiply $b = Ax = [21; 29; 10]$

8. (12 pts) Label each statement as generally TRUE or FALSE.

(a) matrix multiplication is commutative

Answer: false

(b) matrix multiplication is associative

Answer: true

(c) if A is symmetric and B is symmetric, then AB is symmetric

Answer: false, you can see this with a simple 2×2 example

(d) if A and B are both upper triangular, then AB is upper triangular

Answer: true

(e) if x solves a homogenous linear system, then $2x$ solves the system

Answer: true, if $Ax = \vec{0}$ then $A(2x) = \vec{0}$

(f) if A is square, then $\frac{1}{2}(A + A^T)$ is symmetric

Answer: true

9. (10 pts) Three entities transfer gold amongst themselves so that the equilibrium distribution satisfies $Ax = x$ where

$$A = \begin{bmatrix} .1 & .3 & .4 \\ .3 & .2 & .5 \\ .6 & .5 & .1 \end{bmatrix}$$

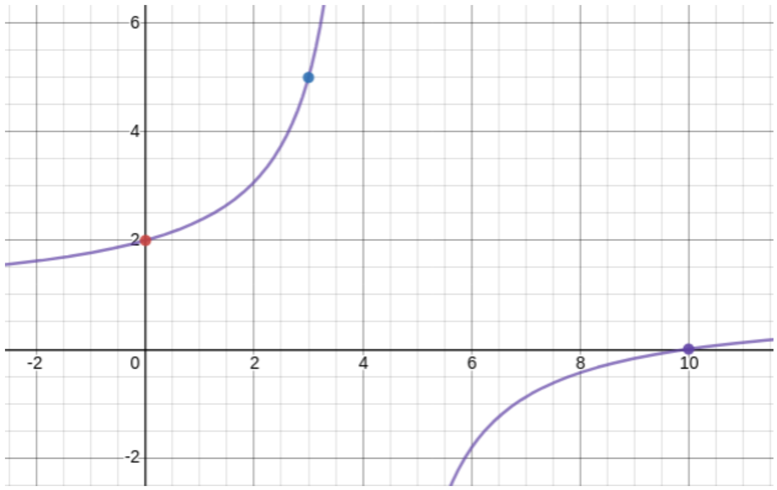
If $x_1 = 47$ and $x_3 = 63$, then find x_2. (Hint: you do not have to row reduce $A - I$).

Answer: the first row of $Ax = x$ says $(.1)(47) + .3x_2 + (.4)(63) = 47$, so $x_2 = 57$

10. (10 pts) Consider designing a function of the form:

$$y = \frac{ax - b}{x - c}$$

so that it passes through the points $(0, 2)$, $(3, 5)$, and $(10, 0)$ as pictured here:



Write the augmented matrix describing a linear system you could solve for a,b, and c. (do not solve).
Answer: plug-in to get $2 = \frac{0-b}{0-c}$, or $b - 2c = 0$
$5 = \frac{3a-b}{3-c}$, or $15 - 5c = 3a - b$
$0 = \frac{10a-b}{10-c}$, or $10a - b = 0$

put those three equations into the matrix:

$$\begin{bmatrix} 0 & -1 & 2 & 0 \\ 3 & -1 & 5 & 15 \\ 10 & -1 & 0 & 0 \end{bmatrix}$$