NAME ______________________________

MATH 208 Test 4, Spring 2020

Directions:

- This test is open book. You may use any resource linked to from the class webpage.
- You must work alone. Do not seek help from any other individual, whether in person or electronically.
- You may use Octave to check your answers, but all work should be done "by hand".
- Use notation conventions as described in class.
- To receive full credit, you must **show all relevant work to completely justify your answer.**
- You have until Tue, Apr 21 at 8am Jeff City time to email me your work. Organize your work clearly.
- 105 points possible, graded out of 100 points.

---

1. (12 pts) Find the area of the quadrilateral with corners at $(0,0)$, $(5,2)$, $(6,7)$, and $(3,9)$. Include a sketch as you show your work.

   Answer: split into two triangles, add areas: $\frac{1}{2}\det\begin{bmatrix}0&5&6\\0&2&7\\1&1&1\end{bmatrix}+\frac{1}{2}|\det\begin{bmatrix}0&3&6\\0&9&7\\1&1&1\end{bmatrix}|=(23+33)/2=28$

2. (10 pts) Find a vector w, with $\|w\|=100$, such that w is orthogonal to both $u=\begin{bmatrix}7\\2\\1\end{bmatrix}$ and $v=\begin{bmatrix}5\\9\\3\end{bmatrix}$.

   Answer: $u\times v=\begin{bmatrix}-3\\-16\\53\end{bmatrix}$, then scale to get $w=\frac{100}{\sqrt{3074}}\begin{bmatrix}-3\\-16\\53\end{bmatrix}$

3. (18 pts) By hand: Find the eigenvalues of this matrix. And for each eigenvalue, find an eigenvector with integer entries.

$$A=\begin{bmatrix}5&0&4\\0&2&0\\3&0&1\end{bmatrix}$$

   Answer: $(5-\lambda)(2-\lambda)(1-\lambda)+3(-4(2-\lambda))=(2-\lambda)(\lambda^2-6\lambda-7)=(2-\lambda)(\lambda-7)(\lambda+1)=0$

   $\lambda=2$, solve $\begin{bmatrix}3&0&4\\0&0&0\\3&0&-1\end{bmatrix}x=0$, so $x=\begin{bmatrix}0\\1\\0\end{bmatrix}$

   $\lambda=7$, solve $\begin{bmatrix}-2&0&4\\0&-5&0\\3&0&-6\end{bmatrix}x=0$, so $x=\begin{bmatrix}2\\0\\1\end{bmatrix}$

   $\lambda=-1$, solve $\begin{bmatrix}6&0&4\\0&3&0\\3&0&2\end{bmatrix}x=0$, so $x=\begin{bmatrix}2\\0\\-3\end{bmatrix}$

4. (12 pts) Let $A=\begin{bmatrix}5&x\\x&2\end{bmatrix}$ be a symmetric matrix. If the largest eigenvalue is 10, find $\det(A)$.

   Answer: method A: the sum of the eigenvalues is the trace, so $10+\lambda_2=7$; therefore $\lambda_2=-3$, and the determinant is the product $(-3)(10) = -30$.

method B: set $(5-\lambda)(2-\lambda) - x^2 = \lambda^2 - 7\lambda + 10 - x^2 = 0$ to get $\lambda_1 = \frac{7+\sqrt{9+4x^2}}{2} = 10$, so $x = \sqrt{40}$. Then $\det(A) = 10 - 40 = -30$.

5. (10 pts) Let A and B be 5×5 matrices. If $\det(A) = 20$ and $\det(B) = 8$, find $\det(-2A^{-1}B^3)$.
   Answer: $(-2)^5 \frac{1}{20}(8)^3 = -819.2$

6. (15 pts) Create a matrix that has these eigenpairs:

$$\{(5, \begin{bmatrix} 4 \\ 11 \end{bmatrix}), (-2, \begin{bmatrix} 3 \\ 8 \end{bmatrix})\}$$

   Answer: $A = V\Lambda V^{-1} = \begin{bmatrix} 4 & 3 \\ 11 & 8 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -8 & 3 \\ 11 & -4 \end{bmatrix} = \begin{bmatrix} -226 & 84 \\ -616 & 229 \end{bmatrix}$

7. (10 pts) Suppose A is 3×3.
   If $\det(A) = 70$, the trace of A is 2, and $A - 8I$ is singular, then find the eigenvalues of A.
   Answer: 8 is an eigenvalue. $8\lambda_2\lambda_3 = 70$ and $8 + \lambda_2 + \lambda_3 = 2$. Solve to get $\lambda = 8, -7/2, -5/2$

8. Let $A = \begin{bmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{bmatrix}$, where $x \geq 0$.

   (a) (6 pts) If $x = 10$, find $\det(A)$. Show your work to calculate the determinant by hand.
   Answer: 9701

   (b) (12 pts) If $\det(A) = 10$, find x.
   (Full credit if you find x EXACTLY, $\frac{1}{2}$ credit if you approximate it to the nearest 0.1.)
   Answer: expand by cofactors to get $x^4 - 3x^2 + 1 = 10$. Then by the QF, $x^2 = \frac{3+\sqrt{45}}{2}$, so $x = \sqrt{\frac{3+\sqrt{45}}{2}} \approx 2.2$.