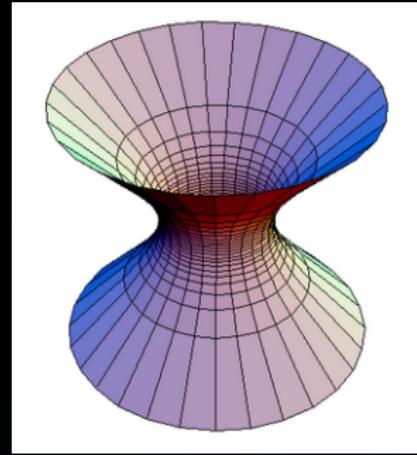


Math 211/213 Calculus III-IV

Kenneth Massey

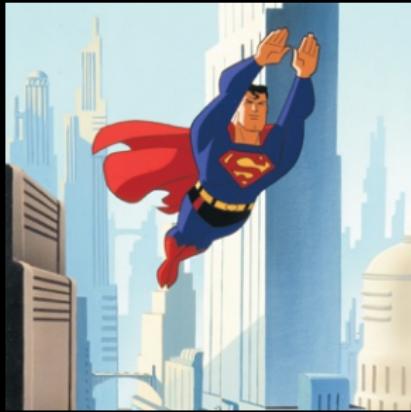
Carson-Newman University

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Directions

You are at the origin and giving directions to the point $(4, 3)$.



1. In Manhattan:
go east 4 blocks, then north 3 blocks.
2. Superman:
angle and distance

Polar Coordinates

Converting between $(x, y)_C$ and $(\theta, r)_P$.

- ▶ $r^2 = x^2 + y^2$
- ▶ $x = r \cos \theta$
- ▶ $y = r \sin \theta$
- ▶ $\tan \theta = \frac{y}{x}$



Aliases

Consider the point $\left(\frac{-\pi}{6}, 8\right)_P$

1. Cartesian coordinates
(unique)
2. List some **aliases**
in polar coordinates.
3. The set of all aliases is
 $\left(\frac{-\pi}{6} + k\pi, (-1)^k 8\right)_P, k \in \mathbb{Z}$



Conversion

When converting to polar,
make sure the quadrants match.

1. $(\frac{\pi}{4}, 6)_P$
2. $(2, 1)_C$
3. $(-1, 1)_C$
4. $(\frac{\pi}{2}, -1)_P$
5. $(\frac{-5\pi}{6}, -3)_P$
6. $(-\sqrt{3}, -1)_C$



Polar Graphs

The graph of an equation/inequality is the set of points that satisfy it. (Geogebra or Desmos)

1. $r = 2$

2. $\theta = 2$

3. $r < 3$

4. $1 \leq r \leq 2$

5. $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$

6. $r = \theta$

7. $r = \frac{1}{\theta}, \theta \in [1, 6\pi]$

8. $r = 2 \sin(\theta)$

9. $r = \sin(2\theta)$

10. $r = \frac{1}{\sin \theta + 2 \cos \theta}$
(convert to Cart.)

11. $r^2 \sin \theta \cos \theta = 1$
(convert to Cart.)

12. $(x - 2)^2 + y^2 = 4$
(convert to polar)

Name that Shape

Substitute and complete the square to identify the graph of $r = a \cos \theta + b \sin \theta$.

$$\begin{aligned} r &= a(x/r) + b(y/r) \\ r^2 &= ax + by \end{aligned}$$

⋮

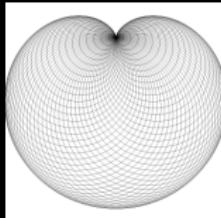


Parameterize

- ▶ $y = x^2$ for $x \in [0, 2]$
just a static path
 x indep., y dep.
- ▶ $x = t, y = t^2$ for $t \in [0, 2]$
dynamic motion along path
 t indep., both x and y dep. (table t, x, y)
- ▶ $x = 3^t, y = 3^{2t}$ for $t \in \mathbb{R}$
- ▶ $x = \sin(t), y = \sin^2(t)$ for $t \in \mathbb{R}$
- ▶ Parameterize $r = \theta^2, \theta \in [0, 2\pi]$.
Then move it north 20, west 10, and
double-time it.



Cardioid



The graph of $r = 1 - \sin(\theta)$ is a [cardioid].

- ▶ Traced once for $\theta \in [0, 2\pi]$.
- ▶ Can think of oriented dynamic motion.
(train's position on a track)
- ▶ Corresponding parametric equations:

$$x = (1 - \sin(t)) \cos(t)$$

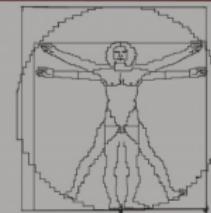
$$y = (1 - \sin(t)) \sin(t)$$

- ▶ move cusp to $(3, 4)$, and $t \in [0, 1]$
no longer a function $r(\theta)$

$$x(t) = (1 - \sin(2\pi t)) \cos(2\pi t) + 3$$

$$y(t) = (1 - \sin(2\pi t)) \sin(2\pi t) + 4$$

Parametric Equations



- ▶ $t \in D$ is the independent parameter in the domain D (think time or theta)
- ▶ $x(t)$ gives dependent x as a function of t
- ▶ $y(t)$ gives dependent y as a function of t
- ▶ as coordinates $(x(t), y(t))$
- ▶ or a vector $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$
- ▶ generalizes $y(x)$ and $r(\theta)$, and to higher dimensional space

Unit Circle



Describe the unit circle

1. implicitly $x^2 + y^2 = 1$
2. two functions $y = \pm\sqrt{1 - x^2}$
3. polar $r = 1$
4. parametrically $[x = \cos(t); y = \sin(t)]$
 - ▶ independent variable $t \in [0, 2\pi]$
 - ▶ initial/terminal point $(0, 1)$
 - ▶ counter-clockwise
 - ▶ angular velocity $\omega = 1$ radian/sec.

How would you change radius? center?

Variations

1. Experiment with software to control
 - ▶ radius
 - ▶ center
 - ▶ period $\frac{2\pi}{\omega}$, frequency $\frac{\omega}{2\pi}$, speed ωr
 - ▶ starting position
 - ▶ clockwise / counter-clockwise
2. ellipse $\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$
3. spiral

Parametric Graphs



- ▶ Any function $y = f(x)$ can be parameterized by $x = t, y = f(t)$.
- ▶ If $x = \cos(t), y = \sin^2(t)$, write y as a function of x and state the domain.
- ▶ GeoGebra: $x = \sin(2t), y = \cos(t)$ for $t \in [0, 2\pi]$. $y = \pm \sqrt{\frac{1 \pm \sqrt{1-x^2}}{2}}$
- ▶ Make table of points to sketch by hand.
e.g. $(\cos(3t), \sin(t))$

Changing Forms

- ▶ parametric $x = \sqrt{t + 1}$, $y = \sqrt{t}$
eliminate parameter
solve for t , then substitute
function $y = \sqrt{x^2 - 1}$
- ▶ implicit $x^2 - y^2 = 1$
parametric $x = \sqrt{t^2 + 1}$, $y = t$
or $x = t$, $y = \sqrt{t^2 - 1}$
or $x = \sec(t)$, $y = \tan(t)$



Examples

Describe the path as a function $y = f(x)$, and state the domain.

$$1. \begin{bmatrix} \sqrt{t} \\ t \end{bmatrix}$$

$$2. \begin{bmatrix} t+1 \\ t^3 \end{bmatrix}$$

$$3. \begin{bmatrix} t^2 \\ t \end{bmatrix} \text{ for } |t| \leq 2$$

$$4. \begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix}$$

Polar to Parametric

Example: $r = \sqrt{\theta}$, $\theta \in [0, \infty)$

- ▶ Often r is written as a function of θ .
- ▶ Corresponding **parametric** equations:

$$x(\theta) = r(\theta) \cos \theta$$

$$y(\theta) = r(\theta) \sin \theta$$

- ▶ Sketch by plotting points.
- ▶ symmetric about the x -axis if $r(\theta)$ is even
- ▶ symmetric about the y -axis if $r(\theta)$ is odd

Lines

1. Parameterize $y = 3 + 2(x - 1)$

- ▶ $x = t, y = 3 + 2(t - 1)$
- ▶ $x = t + 1, y = 3 + 2t$
- ▶ $x = t^2 + 1, y = 3 + 2t^2$
- ▶ $x = \cos(t) + 1, y = 3 + 2\cos(t)$
- ▶ if $y = t$, what is x ?

2. The line segment $(x_0, y_0) \rightarrow (x_1, y_1)$ can be parameterized for $t \in [0, 1]$ by

$$x(t) = x_0 + (x_1 - x_0)t$$

$$y(t) = y_0 + (y_1 - y_0)t$$

Parametric Calculus

Illustrate with $x(t) = \cos(t)$, $y(t) = \sin(2t)$.

- ▶ chain rule $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$, so $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- ▶ point (x_0, y_0) , and slope $m = \frac{dy}{dx}$, the tangent line is $y = y_0 + m(x - x_0)$
- ▶ horizontal tangent: usually $\frac{dy}{dt} = \dot{y} = 0$
- ▶ vertical tangent: usually $\frac{dx}{dt} = \dot{x} = 0$
- ▶ concavity: $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}}$

Linearization

- ▶ At the point $(28, 17)$, $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = -3$.
Estimate the position $\Delta t = .2$ seconds later.
What about $\frac{1}{2}$ second ago ?
- ▶ Consider a polar function $r(\theta)$.
Suppose $r(1) = 8$ and $\frac{dr}{d\theta}\Big|_{\theta=1} = 3$.
 1. Find $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, and $\frac{dy}{dx}$.
 2. Find eqn tangent line where $\theta = 1$.
 3. Estimate the point $(x(1.05), y(1.05))$.

$$\Delta r \approx \frac{dr}{d\theta} \Delta \theta$$

Length of Curves

Segment: distance over time is $\frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t}$

Taking limits, define

- ▶ speed $s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

- ▶ arc length $\mathcal{L} = \int_{t_0}^{t_1} s(t) dt$

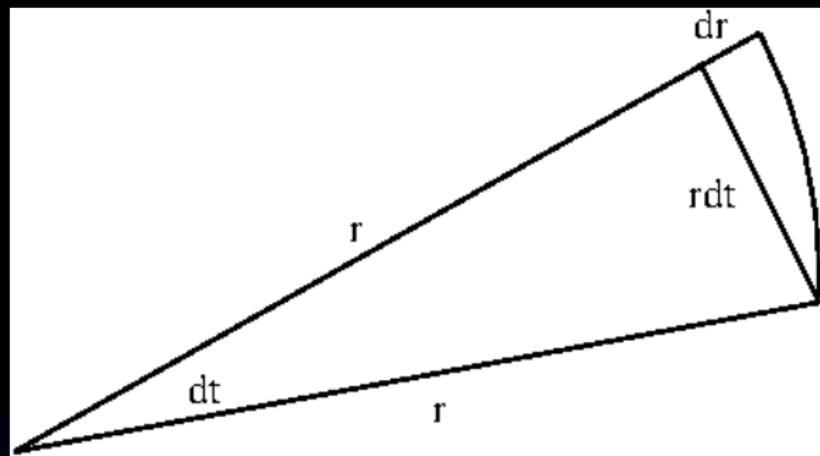
Special cases:

- ▶ $y(x)$, $\mathcal{L} = \int_a^b \sqrt{1 + (dy/dx)^2} dx$

- ▶ $r(\theta)$, $\mathcal{L} = \int_a^b \sqrt{r^2 + (dr/d\theta)^2} d\theta$

Polar Length/Area

Special case formulas can be derived via parametric equations, or by studying:



Verify area and circumference of circle.

Area

- ▶ Take care for \pm area, intersections, overlap.
- ▶ Simple closed curve area:

$$A = \left| \int_{x_0}^{x_1} y dx \right| = \left| \int_{t_0}^{t_1} y \frac{dx}{dt} dt \right|$$

- ▶ Polar with boundary rays: $A = \int_a^b \frac{1}{2} r^2 d\theta$
(integrate circular sectors with area $\frac{\Delta\theta}{2} r^2$)

e.g. between x -axis and $r = \theta$. Show that area works out to $\frac{\pi^3}{6}$ both ways.

Cardioid

$$r = 1 - \sin(\theta)$$

1. Find the tangent line when $\theta = \frac{\pi}{4}$.
2. Find all vertical or horizontal tangents.
3. Find the perimeter.
4. Find the area.

Examples

1. perimeter of region $y = 1 - x^2$, $y = 0$
2. perimeter of $r = 5 \sin(3\theta)$
3. $x = t^2$, $y = t^3 - 4t$, $t \in [0, 2]$
Use the speed (squared) function to find
fastest, slowest. Find path length and area
between curve and x -axis.
4. area of ellipse is πab
5. $[x = e^t \cos(t), y = e^t \sin(t)]$, $t \in [0, T]$
find speed, arc length as functions of T

Fishing

$$t \in [0, 10] \quad x = 4t^2 - 40t + 100$$
$$y = t^3 - 15t^2 + 59t$$



1. table of points, self-intersection
2. find $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$
3. tangent line and concavity where $t = 2$
4. where is tangent either horiz or vertical?
5. where does tangent have slope 1 ?
6. speed when $t = 2$
7. length
8. enclosed area

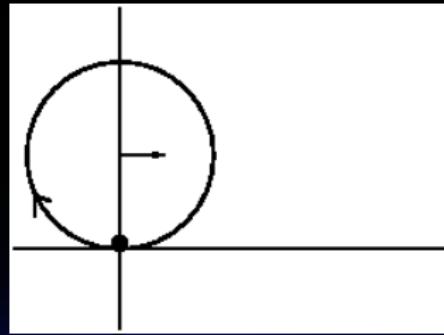
Cycloid

[Path] of point on a rolling wheel

- ▶ center (ht, r)
- ▶ frequency $\frac{h}{2\pi r}$; period $\frac{2\pi r}{h}$ sec.
- ▶ angular velocity $\omega = \frac{h}{r}$ radians per sec.
- ▶ angle $\theta = \omega t = ht/r$
- ▶ point relative to center
 $(-r \sin(\theta), -r \cos(\theta))$
- ▶ parametric equations

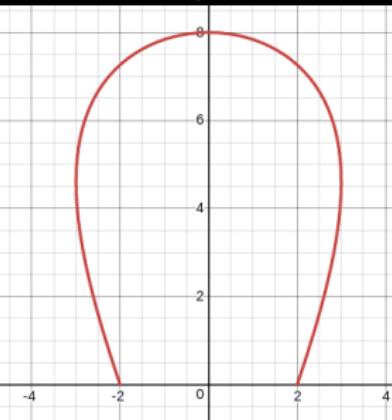
$$x(t) = ht - r \sin(ht/r)$$

$$y(t) = r - r \cos(ht/r)$$
- ▶ arc $8r$, enclosing area $3\pi r^2$ (indep. of h)



Craft Your Own

1. Design parametric equations to produce this horseshoe path for $t \in [0, 10]$.
(sketch x and y as functions of t)
2. Let R be region below curve and above x -axis. Find the perimeter and area of R .
(note $+/-$ 'area' just works!)
3. Tan. line thru $(0, 12)$ with negative slope.



For Everything Else



Find the area of intersection of the two circles:

$$r = 1 \text{ and } r = 2 \cos \theta.$$

- ▶ $2 \left(\frac{\pi}{6} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2} (2 \cos \theta)^2 d\theta \right)$ [WA]
- ▶ $\pi - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2} (2 \cos \theta)^2 - \frac{1}{2} (1)^2 d\theta$ [WA]

What about the area of the circles' union ?