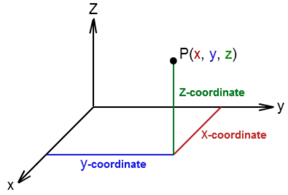
Euclidean Space



- 1D $\mathbb{R}$, 2D $\mathbb{R}^2$, and 3D $\mathbb{R}^3$ space
- Cartesian coordinates (x, y, z)
- cylindrical coordinates (θ, r, z)
- associate each pt. with vector from origin
- vector $\vec{v} = \vec{PQ} = “Q - P”$
  from P (initial point)
  to Q (terminal point)

Example

$P(3,2,5)$ and $Q(5,-1,12)$ can be thought of as points or vectors.

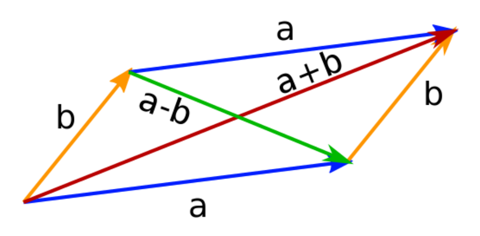
- Sketch $\vec{P}$, $\vec{Q}$, and $\vec{PQ} = [2,-3,7] \in \mathbb{R}^3$
- write as row, column, or in standard basis:
$$\begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2\vec{i} - 3\vec{j} + 7\vec{k}$$
- distance from P to Q is the length of $\vec{PQ}$
$\sqrt{2^2 + 3^2 + 7^2}$

Vectors

- visualize vector as arrow having length and direction
- array of coordinates, e.g. $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$
- $\vec{i}, \vec{j}, \vec{k}$ standard 3D basis, $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$
- the dimension is the number of coordinates

Linear Combinations

vector addition and scalar multiplication maintain nice properties (see book)



Distance

The distance between P and Q is the length/magnitude/norm of the vector $\vec{PQ}$. Generalize the Pythagorean Theorem.

- between $P(2)$ and $Q(5)$ in $\mathbb{R}$
- between $P(2,1)$ and $Q(5,5)$ in $\mathbb{R}^2$
- between $P(2,1,8)$ and $Q(5,5,2)$ in $\mathbb{R}^3$

$$||\vec{PQ}|| = \text{dist}(P, Q) = \sqrt{\sum |q_i - p_i|^2}$$

Euclidean Norm

Definition 1

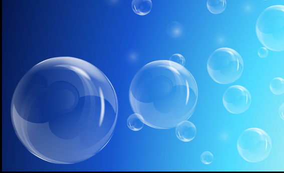
If $\vec{v} \in \mathbb{R}^n$, the Euclidean **norm** $\|\vec{v}\| = \sqrt{\sum_{i=1}^{n} v_i^2}$ measures the vector's length or magnitude.

- triangle inequality $\|\vec{a} + \vec{b}\| \le \|\vec{a}\| + \|\vec{b}\|$
- scalars factor out $\|c\vec{v}\| = |c|\|\vec{v}\|$
- **zero vector** $\vec{0} = [0, 0, 0]$, $\|\vec{0}\| = 0$

Unit Vectors

- a unit vector has $\|\vec{u}\| = 1$
- $\frac{\vec{v}}{\|\vec{v}\|}$ is a unit vector if $\vec{v} \neq \vec{0}$
- Any non-zero vector can be written as length times direction: $\vec{v} = \|\vec{v}\| \frac{\vec{v}}{\|\vec{v}\|}$
- e.g. if $\vec{v} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$, then $\vec{u} = \frac{1}{\sqrt{45}} \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix}$, and $-3\vec{u}$ has length 3 but in the opposite dir.
- in $\mathbb{R}^2$, a unit vector can be written as $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

Spheres

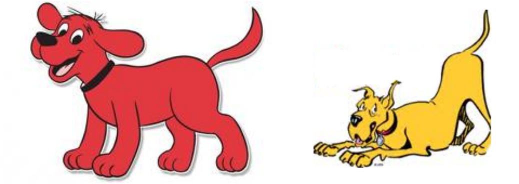


1. Set of all points 1 unit from origin:
   unit sphere - Cartesian and cylindrical.
2. Sphere w. center (x_0, y_0, z_0) and radius R.

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$$

3. centered at $(3, 5, 6)$ and tangent to xy-plane
4. w. diameter $P(1, 5, 3) \to Q(5, -2, 8)$
5. Find center and radius of:
   $x^2 + y^2 + z^2 = 4z - 6x$.

Dog Leash



Taking Clifford and Marmaduke for a walk.

- Clifford pulling 30° west of north.
- Marmaduke pulling northeast.

I am exerting a 50 pound force south so that there is no acceleration of the system.
How much force is each dog exerting?

Right Triangle

Three points $P(-2,5)$, $Q(1,3)$, and $R(5,9)$.

1. Note that $\vec{PQ} + \vec{QR} = \vec{PR}$.
2. Show that $\vec{PQ} \perp \vec{QR}$.
3. Show that $\triangle PQR$ is right using P.T.
4. Find its area.

Dot Product

Definition 2

Given vectors $\vec{a}, \vec{b} \in \mathbb{R}^n$, the dot product is

$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i$$

The dot product of two vectors is a scalar.

Illustration

If $\vec{a} = [1,4,6]$, $\vec{b} = [3,-5,7]$, $\vec{c} = [3,2,1]$, find

1. $\vec{a} \cdot \vec{b} = 25$
2. $\vec{a} \cdot \vec{c} = 17$
3. $2(\vec{a} \cdot \vec{b})$
4. $(2\vec{a}) \cdot \vec{b}$
5. $\vec{a} \cdot (\vec{b} + \vec{c})$
6. $\vec{0} \cdot \vec{a}$
7. $(\vec{a} \cdot \vec{b})\vec{c}$
8. $\vec{a} \cdot \vec{b} \cdot \vec{c}$

Dot Product Properties

- $\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$
- $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ commutative
- $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ distributive
- $(k\vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$ associative
- $\vec{0} \cdot \vec{a} = 0$

Orthogonality

Definition 3

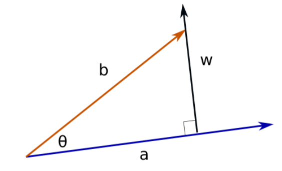
Two vectors are orthogonal (perpendicular, $\perp$) if they satisfy the Pythagorean Theorem:

$$\|\vec{a}\|^2 + \|\vec{b}\|^2 = \|\vec{a} + \vec{b}\|^2$$

Theorem 4

$\vec{a} \perp \vec{b}$ *if and only if* $\vec{a} \cdot \vec{b} = 0$

Orthogonal Decomposition

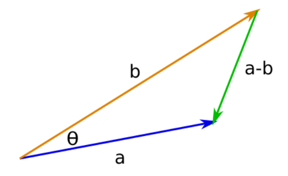


$$\vec{b} = \|\vec{b}\| \cos(\theta) \frac{\vec{a}}{\|\vec{a}\|} + \|\vec{b}\| \sin(\theta) \frac{\vec{w}}{\|\vec{w}\|}$$

Dot product $\vec{a}$ on both sides, noting $\vec{a} \cdot \vec{w} = 0$

- $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$
- $\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \|\vec{b}\|}\right) \in [0, \pi]$

Law of Cosines



- Let θ be the angle between $\vec{a}$ and $\vec{b}$.
- Draw a triangle; note $\vec{a} = \vec{b} + (\vec{a} - \vec{b})$.
- Derive the Law of Cosines, a generalization of the Pythagorean Theorem.

$$\begin{aligned} \|\vec{a} - \vec{b}\|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\vec{a} \cdot \vec{b} \\ &= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\| \cos(\theta) \end{aligned}$$

Finding Angles

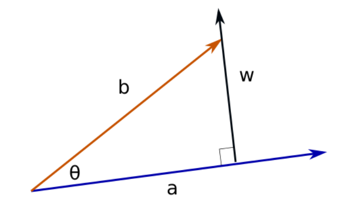
1. Find all three angles of the triangle formed by $P(0,0)$, $Q(3,1)$, and $R(5,4)$.
2. The Great Pyramid of Giza has a square base of side length 756 feet, and was 481 feet tall. Find the angle between two adjacent lateral edges. Then find the area of a triangluar face.

Correlation

- $\vec{1}$ is the vector of ones.
- mean: $\overline{x} = \dfrac{\vec{1} \cdot \vec{x}}{n}$
- Let $\tilde{x} = \vec{x} - \overline{x}\vec{1}$ be the "centered" vector.
- correlation:
  $$\cos(\theta) = \frac{\tilde{M} \cdot \tilde{E}}{\|\tilde{M}\|\|\tilde{E}\|} \in [-1, 1]$$

math	english
24	29
17	20
31	27
23	32
20	22

Projections



$$\vec{b} = \|\vec{b}\|\cos(\theta)\frac{\vec{a}}{\|\vec{a}\|} + \|\vec{b}\|\sin(\theta)\frac{\vec{w}}{\|\vec{w}\|}$$

- the scalar component of $\vec{b}$ along $\vec{a}$ is $\mathrm{comp}_{\vec{a}}\vec{b} = \|\vec{b}\|\cos(\theta) = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|}$
- the orthogonal projection of $\vec{b}$ onto $\vec{a}$ is $\mathrm{proj}_{\vec{a}}\,\vec{b} = \mathrm{comp}_{\vec{a}}\vec{b}\,\frac{\vec{a}}{\|\vec{a}\|} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\vec{a}$

Projections

1. Let $\vec{a} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.
   Compute and illustrate $\mathrm{proj}_{\vec{a}}\, \vec{b}$ and $\mathrm{proj}_{\vec{b}}\, \vec{a}$.
2. Consider a line ℓ that contains P and Q. Show that these expressions give the shortest distance from a point R to ℓ:

$$\|\vec{PR} - \mathrm{proj}_{\vec{PQ}}\, \vec{PR}\| = \sqrt{\|\vec{PR}\|^2 - (\mathrm{comp}_{\vec{PQ}}\, \vec{PR})^2}$$

Work

- force: $\vec{F}$
- displacement: $\vec{D}$
- work: $W = \vec{F} \cdot \vec{D} = \|\vec{F}\|\|\vec{D}\| \cos(\theta)$

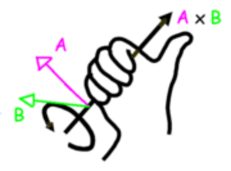
Cross Product

Definition 5

The cross product of two vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$ is

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} \\
&= \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ -(a_1 b_3 - a_3 b_1) \\ a_1 b_2 - a_2 b_1 \end{bmatrix}
\end{aligned}$$

Miracle



- rotate fingers from $\vec{a}$ to $\vec{b}$, then thumb gives direction of $\vec{a} \times \vec{b}$
- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$
- $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin(\theta)$

Example

Let $\vec{a} = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

1. Show $\vec{a} \times \vec{a} = \vec{0}$.
2. Show $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.
3. Show $\vec{a} \perp (\vec{a} \times \vec{b})$.
4. Show $\vec{b} \perp (\vec{a} \times \vec{b})$.
5. Show $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$.

Multiplication Table

	$\vec{i}$	$\vec{j}$	$\vec{k}$
$\vec{i}$	$\vec{0}$	$\vec{k}$	$-\vec{j}$
$\vec{j}$	$-\vec{k}$	$\vec{0}$	$\vec{i}$
$\vec{k}$	$\vec{j}$	$-\vec{i}$	$\vec{0}$

Properties

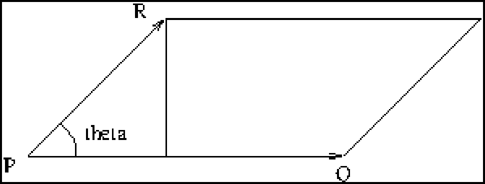
1. $\vec{a} \times \vec{a} = \vec{0}$
2. $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$; anti-commutative
3. $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b})$ for constant c
4. $\vec{a} \times (\vec{b} + \vec{z}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{z})$; distributive

Give an example showing not associative.

Dot vs Cross

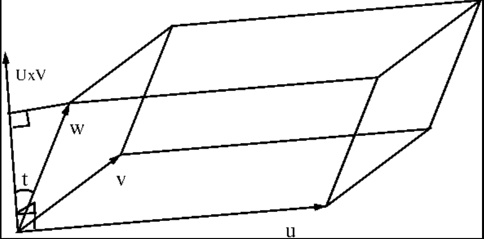
dot product	cross product
defined for $\mathbb{R}^n$	defined only for $\mathbb{R}^3$
$\vec{x} \cdot \vec{y}$ is a scalar	$\vec{x} \times \vec{y}$ is a vector
$\vec{x} \cdot \vec{y} = \lVert\vec{x}\rVert \lVert\vec{y}\rVert \cos\theta$	$\lVert\vec{x} \times \vec{y}\rVert = \lVert\vec{x}\rVert \lVert\vec{y}\rVert \sin\theta$
$\vec{x} \cdot \vec{y} = 0$ implies $\vec{x} \perp \vec{y}$	$\vec{x} \times \vec{y} = \vec{0}$ implies $\vec{x} = c\vec{y}$
$\vec{x} \cdot \vec{x} = \lVert\vec{x}\rVert^2$	$\vec{x} \times \vec{x} = \vec{0}$

Parallelogram



- area of parallelogram defined by P, Q, R
  $\|\vec{PQ}\|\|\vec{PR}\| \sin\theta = \|\vec{PQ} \times \vec{PR}\|$
- $\triangle PQR$ has area $\frac{1}{2}\|\vec{PQ} \times \vec{PR}\|$
- If pts all in $\mathbb{R}^2$, append 0 to extend to $\mathbb{R}^3$.

Parallelpiped



Three vectors $\vec{u}, \vec{v}, \vec{w}$ determine a parallelpiped.

$$\begin{aligned}
\text{volume} &= (\text{area base})(\text{height}) \\
&= \|\vec{u} \times \vec{v}\|(\|\vec{w}\||\cos\theta|) \\
&= \|\vec{u} \times \vec{v}\|\|\vec{w}\|\frac{|\vec{w} \cdot (\vec{u} \times \vec{v})|}{\|\vec{w}\|\|\vec{u} \times \vec{v}\|} \\
&= |\vec{w} \cdot (\vec{u} \times \vec{v})|
\end{aligned}$$

Example

$P(1,2,3)$, $Q(5,7,1)$, $R(-1,4,0)$, $S(2,2,8)$

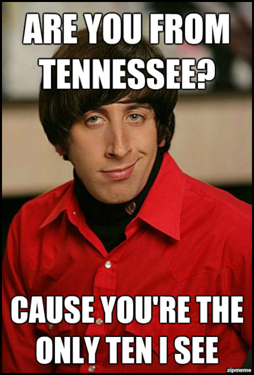
1. Find unit vector $\perp$ to plane PQR.
   - using dot products
   - using cross product
2. Find angle between $\vec{PS}$ and the plane PQR.
3. Find area $\triangle PQR$.
4. Find volume parallelpiped with corners at P, Q, R, S.

Lines in 2D

Consider the line thru $P(1,2)$ and $Q(6,4)$.

- Instead of pt/slope or pt/intercept form, parameterize the line.
- $\vec{\ell}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} t.$
- $x(t) = 1 + 5t$
  $y(t) = 2 + 2t$
- at P when $t = 0$; at Q when $t = 1$
- $t \in \mathbb{R}$ describes infinite linear path
- $t \in [0,1]$ describes line segment

Equation of a Line



Given:

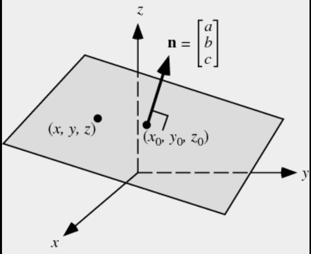
- point on the line (written as vector) $\vec{p}$
- direction vector of the line $\vec{v}$

The equation of the line is:

$$\vec{\ell}(t) = \vec{p} + \vec{v}t$$

Each value of the parameter t identifies a point on the line.

Equation of a Plane



Given:

- point on the plane $\vec{r}_0$
- normal vector $\vec{n}$

If $\vec{r}$ is a point on the plane, then $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

- could collect constants on RHS
- $\vec{n}$ given by coefficients of x, y, z

Examples

1. line thru two points
2. plane determined by three points
3. line $\perp$ to plane
4. line thru point, parallel to another line
5. plane thru point, parallel to another plane
6. plane thru point, $\perp$ to a line
7. Sketch intercepts and find normal vector of $z = 84 - 7x - 4y$.

Intersections

IMAGINE
THE POSSIBILITIES

	line	plane	sphere
line			
plane			
sphere			

Two Lines

Two lines are either:

- parallel (directions are scalar multiples)
- intersecting (pass through same point)
- skew (neither parallel nor intersect)

If you randomly pick two lines, they are most likely skew.

Intersecting Lines

$$\vec{\ell}_1(t_1) = \begin{bmatrix} 3t_1 - 7 \\ t_1 + 1 \\ 2t_1 + 5 \end{bmatrix}, \vec{\ell}_2(t_2) = \begin{bmatrix} 2t_2 + 9 \\ 5t_2 + 15 \\ -t_2 + 11 \end{bmatrix}$$

Set $\vec{\ell}_1(t_1) = \vec{\ell}_2(t_2)$ to find intersection $(5, 5, 13)$.

- when and where
  paths cross, but no collison in time !
- Find the angle of intersection.
- Find eqn of the plane containing these lines.

Skew Lines

$$\vec{\ell}_1(t_1) = \begin{bmatrix} t_1 \\ 1+t_1 \\ 2 \end{bmatrix}, \vec{\ell}_2(t_2) = \begin{bmatrix} t_2 - 1 \\ 2t_2 \\ 1+t_2 \end{bmatrix}$$

- Not parallel since directions $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \neq c \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.
- Do not intersect since this system of three eqns in two unknowns is inconsistent (no solution).

Intersection of Line and Plane

- line $\vec{\ell}(t) = 3(t-2)\vec{i} + (2-5t)\vec{j} + 8t\vec{k}$
- plane $z = 84 - 7x - 4y$

1. Find the point of interesction.
2. Find the acute angle of intersection: let θ be angle between line and normal vector, then find $|90^\circ - \theta|$.
3. Write two lines through the origin that lie parallel to the plane, and make an angle of 30° with each other.

Intersection of Planes

$x - y + 2z = 0$ and $2x + y - z = 1$

1. line of intersection
   - could solve system of 2 eqns, 3 unknowns
   - point: set $z = 0$ and solve for x, y
     direction:
     - find another point
     - $\perp$ to $[1; -1; 2]$ and $[2; 1; -1]$
2. angle between the planes = angle between their normal vectors

Shortest Distance

Draw pictures to illustrate.

1. point Q to plane:
   - if P in plane, find $\|\text{proj}_{\vec{n}} \vec{PQ}\| = \frac{|\vec{PQ} \cdot \vec{n}|}{\|\vec{n}\|}$
2. point Q to line:
   - use a projection
   - $\vec{\ell}(t) = \vec{p} + \vec{v}t$, set $\vec{v} \cdot (\vec{\ell}(t) - \vec{q}) = 0$
3. between two skew lines
   - set $(\vec{p}_1 + \vec{v}_1 t_1) - (\vec{p}_2 + \vec{v}_2 t_1) \perp \vec{v}_1, \vec{v}_2$
   - find component of $\vec{p}_1 - \vec{p}_2$ along $\vec{v}_1 \times \vec{v}_2$