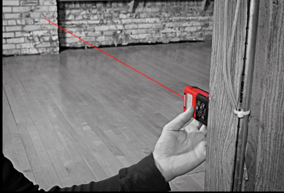
# Example



Consider the solid under the vaulted ceiling $f(x, y) = 5 + \frac{1}{4}xy^2$, and over the rectangular region $R = [1, 5] \times [-2, 4]$ of the $xy$-plane.

1. Sketch.
2. Divide the base into a grid of 6 squares.
3. Estimate the volume using midpoint heights $V \approx 4[f(2, -1) + f(2, 1) + f(2, 3) + f(4, -1) + f(4, 1) + f(4, 3)]$

# Riemann Sums

- recall "area" calculation

$$\sum_{i=1}^{m} f(x_i)\Delta x_i \to \int_a^b f(x)\,dx$$

- partition rectangular base $[a, b] \times [c, d]$ into grid with column base $\Delta A_{ij} = \Delta x_i \Delta y_j$.
- estimate "volume"

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\Delta x_i \Delta y_j$$

# Double Integral over Rectangle

- as $\Delta A_{ij} \to 0$, the limit is: $\iint\limits_R f(x,y)\,dA$

$$= \int_c^d \left[ \int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[ \int_c^d f(x,y)\,dy \right] dx$$

- inside the iterated integral, hold the outer variable constant to compute cross-sectional area of a slice
- if the function is well-behaved (e.g. continuous), Fubini's Theorem says that the order of integration does not matter

# Examples

1. Compute $\iint\limits_R 6x^2 y\,dA$ in both orders; illustrate how constants can be moved.
2. Which order of integration is easier? $\iint\limits_R xe^{xy}\,dA$
3. Evaluate $\int_0^3 \int_{-1}^1 \sqrt{1-x^2}\,dx\,dy$
4. Evaluate $\displaystyle\int_0^{4\pi} \int_1^5 (1+\sin(x))\,y\,dy\,dx$

# Water Bill

Pool $R = [0, 20] \times [0, 17]$, with water at $z = 0$, and bottom steepest descent diagonal from $(0, 0, -2)$ to $(20, 17, -18)$.

- Let $z = -2 + ax + by$, with gradient $\nabla z = [a; b] = c[20; 17]$. Set $-2 + 20a + 17b = -18$ to get $c = \frac{16}{689}$, so $z = -2 - \frac{320}{689}x - \frac{272}{689}y$.
- $f(x, y) = 0 - z = 2 + \frac{320}{689}x + \frac{272}{689}y$ is depth.
- $V = \iint_R f(x, y)\,dA = 3400$ cubic feet
- $A = \iint_R 1\,dA = 340$ square feet
- $\frac{V}{A} = 10$ feet is the average depth

# Area and Volume

- The area of a region $R$ is
  $$A = \iint\limits_R 1 \, dA.$$
- If $f \geq 0$ represents top minus bottom height, the volume is
  $$V = \iint\limits_R f(x, y) \, dA.$$
- The average value of a function is
  $$f_{avg} = \frac{V}{A}$$

# Examples

1. Improper integral of $z = \frac{2y}{x^2}$ over $[1, \infty) \times [0, 3]$
2. Find volume and avg height of solid bounded by (bdb): $|x| = 3$, $|y - 7| = 5$, $z = x^2$ (floor), and $z = 50 + xy$ (ceiling)
3. Set up an integral to find the volume between $f(x, y) = 16 - x^2 - y^2$ and its tangent plane in a unit square region centered around $(2, 1)$. What is the average vertical gap on that region?
4. A rectangular city is $5 \times 8$ kilometers. Find the avg distance to a particular corner.

# Non-Rectangular Region

- Find volume of solid bdb:
  $y = 0, y = 2, x = 0, x = 2, z = 0, z = 4 - x^2$
- Now what if $R$ not rectangular?
  Replace $x = 2$ with $y = x$.
  $$\int_0^2 \int_0^y (4 - x^2)\, dx dy = \int_0^2 \int_x^2 (4 - x^2)\, dy dx$$

# General Regions

- Sketch $R$ in the $xy$-plane.
- Find intersections.
- Draw the more convenient cross-sections.
- Outside variable marches from low to high values.
- Inside variable goes from lower to upper boundary, which may depend on the outside variable.
- Setup: start outside, move in.
- Evaluation: start inside, work out

# Order Matters

1. $R$ bdb triangle $(2, 4)$, $(2, 10)$, and $(6, 10)$
2. $R$ bdb triangle $(0, 4)$, $(2, 10)$, and $(6, 10)$
3. Set-up $\iint_R f(x, y)\,dA$ for $R$ bdb $y = 17 - x^2, y = 5 - 4x$.
4. Change order of integration, then evaluate $\int_0^1 \int_y^1 e^{x^2}\,dxdy$
5. Evaluate $\int_1^e \int_0^{\ln x} y\,dydx$.

Wolfram Alpha:
"integrate y dy dx, y=0 to ln(x), x=1 to e"

# Examples

1. Evaluate $\int_0^\infty \int_0^{e^{-x^2}} 2xy\,dy\,dx$
2. $\iint\limits_R \frac{1}{y}\,dA$ for $R$ bounded below by $y=1$, and above by $|x|y^2 = 1$

# Drone Delivery


Our $5 \times 8$ city has annexed a region adjacent to its southern boundary, bounded by $y = 0$ and $y = cx(x-5)$. The total area of the city is now $90\ km^2$.

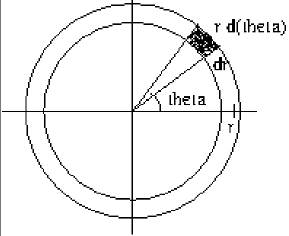
1. Find the perimeter of the city.
2. Walmart is located at $(0, 0)$. Find the average drone flying distance to all points in the city.

# GL Distance

Consider a square $[0, L] \times [0, L]$. Find the average distance to the Gordon line $y = x$.

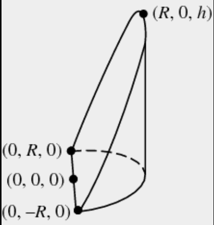
- From a point $(x, y)$ to the closest GL pt $(a, a)$, we know $\begin{bmatrix} x - a \\ y - a \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$, so $a = \frac{x+y}{2}$, and the distance is $\frac{1}{\sqrt{2}}|x - y|$.
- $\displaystyle\int_0^L \int_0^x \frac{1}{\sqrt{2}}(x - y)\,dy\,dx = \frac{L^3}{6\sqrt{2}}$
- Divide by $L^2/2$ to get the average distance of $\frac{L}{3\sqrt{2}}$.

# Polar




- The area of the shaded sector approximately (arc length) $\times$ (width) $dA = r dr d\theta$
- change of variables: $$\iint_R f(x, y)\, dx dy = \iint_R f(r\cos\theta, r\sin\theta)\, r dr d\theta$$
- Note that $A = \iint_R r dr d\theta = \int \frac{1}{2} r^2 d\theta$.

# Cylindrical Wedge




Find the volume of the wedge bounded by $z = 0$, $z = x$, and $x^2 + y^2 = 4$.

- Cartesian: $2 \int_0^2 \int_0^{\sqrt{4-y^2}} x\,dx\,dy$
- polar: $\int_{-\pi/2}^{\pi/2} \int_0^2 r^2 \cos(\theta)\,dr\,d\theta$

Generalize for radius $R$ and height $h$.

# Examples

1. area of a circle: $\int_0^{2\pi} \int_0^a r dr d\theta = \pi a^2$
2. volume of a sphere:
   $2 \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2} r dr d\theta = \frac{4}{3}\pi a^3$
3. avg height of hemisphere dome
4. vol. solid bdb $z = 4 - x^2 - y^2$ and $z = 0$
   $\int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta = 8\pi$
5. integrate $z = e^{x^2+y^2}$ over first loop of
   $r = \sin(2\theta)$
6. average distance to point on boundary of circle of radius 5