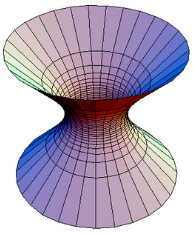
Math 211/213
Calculus III-IV



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Directions

You are at the origin and giving directions to the point $(4, 3)$.

1. In Manhattan:
   go east 4 blocks, then north 3 blocks.
2. Superman:
   angle and distance

Polar Coordinates

Converting between $(x, y)_C$ and $(\theta, r)_P$.

- $r^2 = x^2 + y^2$
- $x = r\cos\theta$
- $y = r\sin\theta$
- $\tan\theta = \frac{y}{x}$

Aliases

Consider the point $\left(\frac{-\pi}{6}, 8\right)_P$

1. Cartesian coordinates (unique)
2. List some aliases in polar coordinates.
3. The set of all aliases is $\left(\frac{-\pi}{6} + k\pi, (-1)^k 8\right)_P, k \in \mathbb{Z}$

Conversion

When converting to polar, make sure the quadrants match.

1. $(\frac{\pi}{4}, 6)_P$
2. $(2, 1)_C$
3. $(-1, 1)_C$
4. $(\frac{\pi}{2}, -1)_P$
5. $(\frac{-5\pi}{6}, -3)_P$
6. $(-\sqrt{3}, -1)_C$

Polar Graphs

The graph of an equation/inequality is the set of points that satisfy it. (Geogebra or Desmos)

1. $r = 2$
2. $\theta = 2$
3. $r < 3$
4. $1 \le r \le 2$
5. $\frac{\pi}{6} \le \theta \le \frac{\pi}{3}$
6. $r = \theta$
7. $r = \frac{1}{\theta}, \theta \in [1, 6\pi]$
8. $r = 2\sin(\theta)$
9. $r = \sin(2\theta)$
10. $r = \frac{1}{\sin\theta + 2\cos\theta}$ (convert to Cart.)
11. $r^2 \sin\theta \cos\theta = 1$ (convert to Cart.)
12. $(x-2)^2 + y^2 = 4$ (convert to polar)

Name that Shape

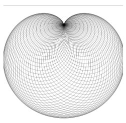
Substitute and complete the square to identify the graph of $r = a\cos\theta + b\sin\theta$.

$$\begin{aligned} r &= a(x/r) + b(y/r) \\ r^2 &= ax + by \\ &\vdots \end{aligned}$$

Parameterize

- $y = x^2$ for $x \in [0, 2]$
  just a static path
  x indep., y dep.
- $x = t, y = t^2$ for $t \in [0, 2]$
  dynamic motion along path
  t indep., both x and y dep. (table t, x, y)
- $x = 3^t, y = 3^{2t}$ for $t \in \mathbb{R}$
- $x = \sin(t), y = \sin^2(t)$ for $t \in \mathbb{R}$
- Parameterize $r = \theta^2, \theta \in [0, 2\pi]$.
  Then move it north 20, west 10, and double-time it.

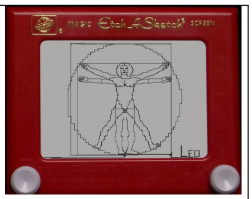
Cardioid



The graph of $r = 1 - \sin(\theta)$ is a [cardioid].

- Traced once for $\theta \in [0, 2\pi]$.
- Can think of oriented dynamic motion.
  (train's position on a track)
- Corresponding parametric equations:
  $x = (1 - \sin(t))\cos(t)$
  $y = (1 - \sin(t))\sin(t)$
- move cusp to $(3, 4)$, and $t \in [0, 1]$
  no longer a function $r(\theta)$
  $x(t) = (1 - \sin(2\pi t))\cos(2\pi t) + 3$
  $y(t) = (1 - \sin(2\pi t))\cos(2\pi t) + 4$

Parametric Equations



- $t \in D$ is the independent parameter in the domain D (think time or theta)
- $x(t)$ gives dependent x as a function of t
- $y(t)$ gives dependent y as a function of t
- as coordinates $(x(t), y(t))$
- or a vector $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$
- generalizes $y(x)$ and $r(\theta)$, and to higher dimensional space

Unit Circle

Describe the unit circle

1. implicitly $x^2 + y^2 = 1$
2. two functions $y = \pm\sqrt{1 - x^2}$
3. polar $r = 1$
4. parametrically $[x = \cos(t); y = \sin(t)]$
   - independent variable $t \in [0, 2\pi]$
   - initial/terminal point $(0, 1)$
   - counter-clockwise
   - angular velocity $\omega = 1$ radian/sec.

How would you change radius? center?

Variations

1. Experiment with software to control
   - radius
   - center
   - period $\frac{2\pi}{\omega}$, frequency $\frac{\omega}{2\pi}$, speed ωr
   - starting position
   - clockwise / counter-clockwise
2. ellipse $\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$
3. spiral

Parametric Graphs

- Any function $y = f(x)$ can be parameterized by $x = t, y = f(t)$.
- If $x = \cos(t), y = \sin^2(t)$, write y as a function of x and state the domain.
- GeoGebra: $x = \sin(2t), y = \cos(t)$ for $t \in [0, 2\pi]$. $y = \pm\sqrt{\frac{1 \pm \sqrt{1-x^2}}{2}}$
- Make table of points to sketch by hand. e.g. $(\cos(3t), \sin(t))$

Changing Forms

- parametric $x = \sqrt{t+1}, y = \sqrt{t}$
  eliminate parameter
  solve for t, then substitute
  function $y = \sqrt{x^2 - 1}$
- implicit $x^2 - y^2 = 1$
  parametric $x = \sqrt{t^2+1}, y = t$
  or $x = t, y = \sqrt{t^2 - 1}$
  or $x = \sec(t)$, $y = \tan(t)$

Examples

Describe the path as a function $y = f(x)$, and state the domain.

1. $\begin{bmatrix} \sqrt{t} \\ t \end{bmatrix}$
2. $\begin{bmatrix} t+1 \\ t^3 \end{bmatrix}$
3. $\begin{bmatrix} t^2 \\ t \end{bmatrix}$ for $|t| \leq 2$
4. $\begin{bmatrix} \cos(t^2) \\ \sin(t^2) \end{bmatrix}$

Polar to Parametric

Example: $r = \sqrt{\theta}, \quad \theta \in [0, \infty)$

- Often r is written as a function of θ.
- Corresponding parametric equations:
  $x(\theta) = r(\theta)\cos\theta$
  $y(\theta) = r(\theta)\sin\theta$
- Sketch by plotting points.
- symmetric about the x-axis if $r(\theta)$ is even
- symmetric about the y-axis if $r(\theta)$ is odd

Lines

1. Parameterize $y = 3 + 2(x - 1)$
   - $x = t$, $y = 3 + 2(t - 1)$
   - $x = t + 1$, $y = 3 + 2t$
   - $x = t^2 + 1$, $y = 3 + 2t^2$
   - $x = \cos(t) + 1$, $y = 3 + 2\cos(t)$
   - if $y = t$, what is x?
2. The line segment $(x_0, y_0) \longrightarrow (x_1, y_1)$ can be parameterized for $t \in [0, 1]$ by

$$\begin{aligned} x(t) &= x_0 + (x_1 - x_0)t \\ y(t) &= y_0 + (y_1 - y_0)t \end{aligned}$$

Parametric Calculus

Illustrate with $x(t) = \cos(t), y(t) = \sin(2t)$.

- chain rule $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, so $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- point (x_0, y_0), and slope $m = \frac{dy}{dx}$, the tangent line is $y = y_0 + m(x - x_0)$
- horizontal tangent: usually $\frac{dy}{dt} = \dot{y} = 0$
- vertical tangent: usually $\frac{dx}{dt} = \dot{x} = 0$
- concavity: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Linearization

- At the point $(28, 17)$, $\frac{dx}{dt} = 5$ and $\frac{dy}{dt} = -3$.
  Estimate the position $\Delta t = .2$ seconds later.
  What about $\frac{1}{2}$ second ago ?
- Consider a polar function $r(\theta)$.
  Suppose $r(1) = 8$ and $\frac{dr}{d\theta}\Big|_{\theta=1} = 3$.
  - Find $\frac{dx}{d\theta}$, $\frac{dy}{d\theta}$, and $\frac{dy}{dx}$.
  - Find eqn tangent line where $\theta = 1$.
  - Estimate the point $(x(1.05), y(1.05))$.
  - $\Delta r \approx \frac{dr}{d\theta}\Delta\theta$

Length of Curves

Segment: distance over time is $\dfrac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t}$

Taking limits, define

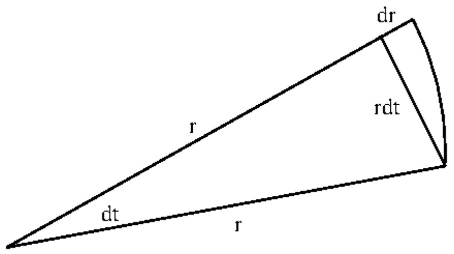
- speed $s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
- arc length $\mathcal{L} = \displaystyle\int_{t_0}^{t_1} s(t)\,dt$

Special cases:

- $y(x)$, $\mathcal{L} = \displaystyle\int_a^b \sqrt{1 + (dy/dx)^2}\,dx$
- $r(\theta)$, $\mathcal{L} = \displaystyle\int_a^b \sqrt{r^2 + (dr/d\theta)^2}\,d\theta$

Polar Length/Area

Special case formulas can be derived via parametric equations, or by studying:



Verify area and circumference of circle.

Area

- Take care for $\pm$ area, intersections, overlap.
- Simple closed curve area:
  $$A = \left| \int_{x_0}^{x_1} y dx \right| = \left| \int_{t_0}^{t_1} y \frac{dx}{dt} dt \right|$$
- Polar with boundary rays: $A = \int_a^b \frac{1}{2} r^2 d\theta$
  (integrate circular sectors with area $\frac{\Delta\theta}{2} r^2$)

e.g. between x-axis and $r = \theta$. Show that area works out to $\frac{\pi^3}{6}$ both ways.

Cardioid

$r = 1 - \sin(\theta)$

1. Find the tangent line when $\theta = \frac{\pi}{4}$.
2. Find all vertical or horizontal tangents.
3. Find the perimeter.
4. Find the area.

Examples

1. perimeter of region bdb $y = 1 - x^2$, $y = 0$
2. perimeter of $r = 5\sin(3\theta)$
3. $x = t^2, y = t^3 - 4t$, $t \in [0, 2]$
   Use the speed (squared) function to find fastest, slowest. Find path length and area between curve and x-axis.
4. area of ellipse is πab
5. $[x = e^t \cos(t), y = e^t \sin(t)]$, $t \in [0, T]$
   find speed, arc length as functions of T

Fishing

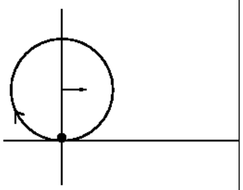
$t \in [0, 10]$ $\quad x = 4t^2 - 40t + 100$
$\quad y = t^3 - 15t^2 + 59t$

1. table of points, self-intersection
2. find $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dy}{dx}$, and $\frac{d^2y}{dx^2}$
3. tangent line and concavity where $t = 2$
4. where is tangent either horiz or vertical?
5. where does tangent have slope 1 ?
6. speed when $t = 2$
7. length
8. enclosed area

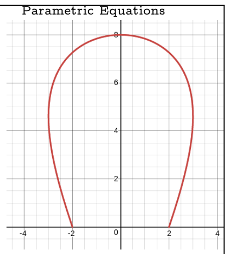
Cycloid

[Path] of point on a rolling wheel

- center (ht, r)
- frequency $\frac{h}{2\pi r}$; period $\frac{2\pi r}{h}$ sec.
- angular velocity $\omega = \frac{h}{r}$ radians per sec.
- angle $\theta = \omega t = ht/r$
- point relative to center
  $(-r\sin(\theta), -r\cos(\theta))$
- parametric equations
  $x(t) = ht - r\sin(ht/r)$
  $y(t) = r - r\cos(ht/r)$
- arc $8r$, enclosing area $3\pi r^2$ (indep. of h)



Craft Your Own



1. Design parametric equations to produce this horseshoe path for $t \in [0, 10]$.
   (sketch x and y as functions of t)
2. Let R be region below curve and above x-axis. Find the perimeter and area of R.
   (note $+/-$ 'area' just works!)
3. Tan. line thru $(0, 12)$ with negative slope.

For Everything Else

Find the area of intersection of the two circles: $r = 1$ and $r = 2\cos\theta$.

- $2\left(\frac{\pi}{6} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(2\cos\theta)^2 d\theta\right)$ [WA]
- $\pi - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(2\cos\theta)^2 - \frac{1}{2}(1)^2 d\theta$ [WA]

What about the area of the circles' union ?