Functions

Input independent variable(s),
output dependent variable(s).

- $y = f(x) = x^2$
  1 input, 1 output (static curve in $\mathbb{R}^2$)
- $x = \cos(t), y = \sin(t)$
  1 input, 2 outputs (dynamic curve in $\mathbb{R}^2$)
- $z = f(x, y) = x^2 + y^2$
  2 inputs, 1 output (surface in $\mathbb{R}^3$)
- $F(x, y) = [x; x + y]$
  2 inputs, 2 outputs (vector field in $\mathbb{R}^2$)

Vector-Valued Function

Consider the line: $\vec{r}(t) = \begin{bmatrix} 1+2t \\ 5t \\ 8-3t \end{bmatrix}$, $t \in \mathbb{R}$

- vector-valued function 1 input, 3 outputs
- parameter t
- output is the "position" at the given "time"
- make a table
- describes a dynamic path
- parametric equations for each coordinate
- what if restrict domain to $t \in [0, 2]$?

Swinging Arm Multi-variable Intro

- area table vs formula $A(a, b) = \frac{1}{3} b^{3/2} a^{-1/2}$
- a and b are independent
  variables/parameters (dials)
- A depends on a and b
- domain $b \leq 36a$
- range $A \geq 0$
- partial derivatives $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$
  are ratios of change in A
  to changes in a or b respectively
- $A(2.03, 27.86) \approx A(2, 28) + \frac{\partial A}{\partial a} \Delta a + \frac{\partial A}{\partial b} \Delta b$

Distance to Hospital

- city boundary $[0, 10] \times [-3, 5]$
- hospital at $H(2, 0)$
- $D(x, y) = \sqrt{(x-2)^2 + y^2}$
- find the range
- visualize as a surface: a cone
- "squared distance": a paraboloid

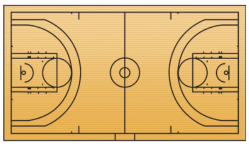
Multi-Variable Function

Consider the plane:

$z(x, y) = 5 + 2x + 7(y-3)$, $(x, y) \in \mathbb{R}^2$

- multi-variable function 2 inputs, 1 output
- given x, y location, gives the "elevation"
- make a table
- describes a surface
- sketch region where x, y, z are all ≥ 0

Domain/Range



- independent variable(s): inputs
- dependent variable(s): outputs
- domain: set of possible inputs
- range: set of possible outputs

Non-flat Examples

1. An object follows a sine curve from $(0, 0)$ when $t = 0$ to $(2, 1)$ when $t = 5$.
   Make a table. Find parametric equations for x and y.
2. Sketch surface $z = f(x, y) = 36 - 4x^2 - y^2$, such that $z \geq 0$.
   What do the cross-sections look like ?

Visualizing

- vector-valued function
  - trajectory of an object moving in space
  - separate parametric equations $(x(t), y(t), z(t))$ can be plotted with respect to t
- multi-variable function
  - surface plot
  - contours
  - heat map

Also, a table of values may be more insightful than a formula, which might not be available.

Contours

A contour, (level or iso curve) is the solution set of $f(x, y) = c$, for a constant c.

- weather iso-bars (barometric pressure), or iso-therms
- topographical maps
- walk along contour - elevation constant
- contours corresponding to distinct levels cannot cross
- contrast contours of a cone and a paraboloid

More Examples

1. $\vec{r}(t) = [\cos(t); \sin(t); t]$, $t \in [0, \infty)$
2. Predator-prey (e.g. wolf-rabbit) state space path; somewhat cyclical, but chaotic.
3. $P(r, m) = \frac{1000r/12}{1-(1+r/12)^{-m}}$
(monthly car payment if borrow 1000 dollars at annual rate r for m months)

Surfaces

- The graph of an equation is the set of points that satisfy it.
- in $\mathbb{R}^2$, graph the "curves"
$y = 1$, $y = x$, $y = x^2$, $x^2 + y^2 = 1$
- in $\mathbb{R}^3$, graph the surfaces:
  - $z = 1$
  - $3x + 4y + z = 24$
  - $z = x^2 + y^2$
  - $z = \sqrt{x^2 + y^2}$
  - $x^2 + y^2 + z^2 = 1$

Multivariable Function Examples

1. $V(r, h) = \pi r^2 h$ (volume of cylinder)
2. $d(h, r) = \sqrt{h^2 + 2rh}$ (distance to horizon from height h on sphere radius r)
3. $F = \frac{Gm_1m_2}{r^2}$ (gravitational force)
4. $z = 7xye^{-(x^2+y^2)}$ (two hills and two valleys)
5. $f(x, y) = x^2 - y^2$ (saddle point)
6. $z = 1 - |x + y| - |x - y|$ (pyramid)

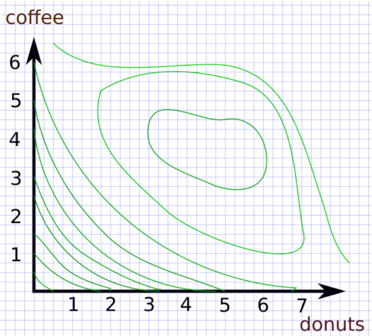
Examples

Find domain, range, and contours.
Visualize surface with software.

1. $z(x, y) = \sqrt{36 - 4x^2 - 9y^2}$.
2. $z = \sqrt{x^2 + y^2}$
3. $z = \ln(xy)$
4. $z = x^2 - y^2$
5. $f(x, y) = e^{-x^2 y}$
6. $f(x, y) = \frac{3x - y}{x^2 + 2y^2 + 1}$

Economics

"indifference curves" represent constant utility (benefit,enjoyment) if trade-offs can be made



Limits

Definition 1

If $|f(x, y) - L|$ can be made arbitrarily small by chosing (x, y) "close" enough to (a, b), then:

$$\lim_{x,y \to a,b} f(x, y) = L$$

Continuity

Definition 2

f is continuous at (a, b) if
$\lim_{x,y \to a,b} f(x, y) = f(a, b)$

Polynomials, rational, trig, logs, exponential functions are continuous *on their domains*.
So just plug in to find the limit.

For example, $z = \ln(x + y^2)$ is continuous on the domain $x > -y^2$.

Approach

For the limit to exist, the answer must be the same no matter how you approach.

- In functions of one variable, can approach from left or right.
- If multi-variable, then can approach from infinitely many directions and paths.

Approach

Use polar coordinates to investigate these limits.

1. $\lim_{x,y \to 0,0} \frac{x^3}{x^2 + y^2} = \lim_{r \to 0} r \cos^3(\theta) = 0$
2. $\lim_{x,y \to 0,0} \frac{xy}{x^2 + y^2}$
3. $\lim_{x,y \to 0,0} \frac{xy^2}{x^2 + y^4}$

GOF

$$g(x, y) = \frac{1}{1 + e^{-u}} \qquad u = \frac{x - y}{2\sqrt{x + y}}$$

x	y	u	g
10	0	1.58	.83
50	40	.53	.63
31	30	.06	.52
42	3	2.91	.95
49	3	3.19	.96
81	0	4.50	.99

Rate of Change

Temperature function grid:

4	46	47	44	46	50
3	33	34	36	37	41
2	26	27	29	30	31
1	23	22	24	25	27
0	22	21	20	21	24
y \ x	0	1	2	3	4

- slope in W-E direction $f_x(2, 2) \approx \frac{30-27}{3-1} = 1.5$
- slope in S-N direction $f_y(2, 2) \approx \frac{36-24}{3-1} = 6$
- slope in NE direction? (rise over run)

Definition 3

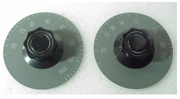
Given $z = f(x, y)$, let $h = \Delta x$.
The (finite) difference quotient w.r.t. x

- centered
$$\frac{\Delta z}{\Delta x} = \frac{f(x + \frac{1}{2}h, y) - f(x - \frac{1}{2}h, y)}{h}$$
- forward
$$\frac{\Delta z}{\Delta x} = \frac{f(x + h, y) - f(x, y)}{h}$$

Definition 4
If the limit exists, the partial derivative of f w.r.t. x is denoted

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = D_x f = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

- all but one indep variable held constant
- slope of surface as you move parallel to axis
- instantaneous rate of change $\frac{\Delta z}{\Delta x} \approx \frac{\partial z}{\partial x}$
- given contour plot, which bigger $|f_x|$ or $|f_y|$?



Calculating Partial Derivatives

1. $\frac{d}{dx}(x^2+1)^4$
2. $\frac{d}{dx}(x^2+2)^4$
3. $\frac{d}{dx}(x^2+3)^4$
4. $\frac{d}{dx}(x^2-1)^4$
5. $\frac{\partial}{\partial x}(x^2+y)^4$
6. $\frac{\partial}{\partial x}(x^2+\cos(y))^4$

- Differentiate w.r.t. one variable; treat all others as constants.
- Use derivative rules (power, product, chain, etc).

Examples

1. $z = f(x,y) = \sqrt{3x^2 + e^y}$; at the pt $(1,0,2)$
   - Estimate $\frac{\partial z}{\partial x} \approx \frac{\Delta z}{\Delta x}$ using nearby points and the difference quotient.
   - Estimate $\frac{\partial z}{\partial y}(1,0) = \left.\frac{\partial z}{\partial y}\right|_{(1,0)}$
   - Use derivative rules to find the partial derivative functions.
2. Find partial derivatives of $z = 3x^2y + \frac{y}{x} + y^4$.
3. $\frac{\partial}{\partial y}\left[y^2(e^x + x^2y)^3\right]$

Tangent Plane

Suppose $f(5,8) = 40$ and $\nabla f(5,8) = [7,3]$.

- point $(5,8,40)$
- normal vector $\begin{bmatrix}1\\0\\f_x\end{bmatrix} \times \begin{bmatrix}0\\1\\f_y\end{bmatrix} = \begin{bmatrix}-7\\-3\\1\end{bmatrix}$

The tangent plane or "linearization" is
$-7(x-5) - 3(y-8) + 1(z-40) = 0$
or rearranging:
$z = 40 + 7(x-5) + 3(y-8)$

Tangent Plane

- $f(x) = x^2$, $f'(x) = 2x$,
  at $(3,9)$ slope 6,
  tangent line $y = 9 + 6(x-3)$
- Follow the same pattern for functions with more than one independent variable.
- $f(x,y) = x^2y^3$, $\nabla f(x,y) = [2xy^3, 3x^2y^2]$
  at $(3,1,9)$, slopes $\nabla f(3,1) = [6,27]$,
  tangent plane $z = 9 + 6(x-3) + 27(y-1)$
- Zoom in enough, and the tangent plane is indistinguishable from the graph's surface.

Differentials

- Intuitively, $f(x,y)$ is differentiable at a pt. if, as you zoom in, the surface becomes flat and coincides with the tangent plane.
- Theorem: if $f(x,y)$ is 1st order smooth, then it is differentiable.
- increment: actual change
  $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$
- differential: approximate change
  using tangent plane as proxy
  $dx = \Delta x$, $dy = \Delta y$
  $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

Linearization

The linearization of f at (x_0, y_0) is equal to the starting function value plus the differential.

$$\begin{aligned}\ell(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) \\ &\qquad + f_y(x_0,y_0)(y-y_0) \\ &= f(x_0,y_0) + \nabla f(x_0,y_0)\cdot\begin{bmatrix}\Delta x\\ \Delta y\end{bmatrix} \\ &= f(x_0,y_0) + dz\end{aligned}$$

Example

Consider $f(x,y) = x^2y$ at $(3,2)$.

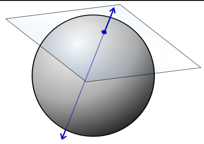
- Contour at $(3,2)$ also passes thru $(x, 50)$.
- $f(3,2) = 18$ and $\nabla f(3,2) = [12, 9]$
- linearization (tangent plane) $z = 18 + 12(x-3) + 9(y-2)$
- differential $dz = 12dx + 9dy$
- Small move to $(3.04, 1.97)$
  - $dz = 12(.04) + 9(-.03) = .21$
  - $\Delta z = f(3.04, 1.97) - f(3,2) = .20595$
  - $f(3.04, 1.97) \approx \ell(3.04, 1.97) = 18.21$

Preserved Information

Suppose the plane $z = 7 - 3(x+4) + 5(y-2)$ is tangent to $f(x,y)$.

1. Point of tangency $(-4, 2, 7)$.
2. Normal vector $\vec{n} = [-3; 5; -1]$
3. Gradient $\nabla f(-4,2) = \begin{bmatrix}-3\\ 5\end{bmatrix}$.
4. Normal line $\ell(t) = \begin{bmatrix}-4\\ 2\\ 7\end{bmatrix} + \begin{bmatrix}-3\\ 5\\ -1\end{bmatrix} t$
5. Estimate $f(-4.13, 2.04)$ $\approx 7 - 3(-.13) + 5(.04) = 7.59$

Normal Line



Consider the paraboloid $f(x, y) = 1 + x^2 + 4y^2$.

1. point on surface $P(3, 1, z)$
2. find gradient
3. find the linearization of f
4. find line thru P normal to surface
5. where would it puncture the surface again?
   at what angle ?

Directional Derivative

Given gradient:
slope if you walk E,W,N,S,NE ?

If z differentiable, use tangent plane to find slope in direction of $\vec{v} = \begin{bmatrix} dx \\ dy \end{bmatrix}$.

$$D_{\vec{v}} z = \frac{\text{rise}}{\text{run}} = \frac{dz}{\|\vec{v}\|} = \frac{\nabla z \cdot \vec{v}}{\|\vec{v}\|}$$

Derivatives of Derivatives

$$f(x, y) = 7x^3 y + \frac{5}{y}$$

- gradient vector
  $$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} 21x^2 y \\ 7x^3 - 5y^{-2} \end{bmatrix}$$
- Hessian matrix $H = \begin{bmatrix} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \end{bmatrix}$

$$= \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 42xy & 21x^2 \\ 21x^2 & 10y^{-3} \end{bmatrix}$$

Smooth

Definition 5
f is kth order smooth if f and all partial derivatives up to order k exist and are continuous.
We may say f is in the class of C^k functions.

No corners, edges, holes, cusps, jumps, singularities, etc.

Higher Order Derivatives

1. f_{xx} and f_{yy} are concavity E/W and N/S
2. Clairaut's Theorem (equality of mixed partials): $f_{xy} = f_{yx}$ if f is 2nd order "smooth"
3. $f_{xyx} = (f_{yx})_x = (f_{xy})_x = f_{xxy}$, etc.
4. $z = f(x, y) = 3x^2y + x(x - y)^7$ find $\nabla z|_{(2,1)}$ and $H_f(2, 1)$; evaluated at $(2, 1)$
5. Compute all first and second order partials $f(x, y, z) = x^2yz - 2zx + y^4z^2$.
6. $PV = cT$; find all second derivatives of P

Partial Differential Equations

1. A function is called "harmonic" if it satisfies the Laplace equation $u_{xx} + u_{yy} = 0$. Show that $u = e^x \sin y$ is harmonic.
2. Show that $u = e^{-x} \sin(t - x)$ satisfies the heat equation $u_t = ku_{xx}$.

Example

Consider the surface of $f(x,y) = 16 - x^2 - y^2$.

1. Find slope in direction $\vec{v} = [2;5]$.
   $D_{\vec{v}}f = \dfrac{[-2x,-2y]\cdot[2;5]}{\sqrt{29}} = \dfrac{-4x-10y}{\sqrt{29}}$
2. Evaluate at $(1,3)$ to get $D_{\vec{v}}f(1,3) = \frac{-34}{\sqrt{29}}$.
3. Find concavity.
   $D_{\vec{v}}\left(\dfrac{-4x-10y}{\sqrt{29}}\right) = \left[\dfrac{-4}{\sqrt{29}}, \dfrac{-10}{\sqrt{29}}\right] \cdot \dfrac{[2;5]}{\sqrt{29}} = -2$

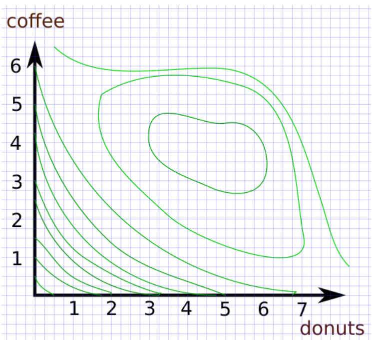
Steepest Ascent

$D_{\vec{v}}f = \frac{\|\nabla f\|\|\vec{v}\|\cos\theta}{\|\vec{v}\|} = \|\nabla f\|\cos\theta.$

- steepest ascent direction $\theta = 0$, so $\vec{v} = \nabla f$
- steepest descent direction $-\nabla f$
- The steepest ascent slope is $\|\nabla f\|$.
- contour direction satisfies $\nabla f \cdot \vec{v} = 0$, so ∇f is $\perp$ to contours

More Please

Sketch gradients at $(1,1)$, $(1,4)$, and $(4,1)$.



Example

Suppose you are on a rolling hillside, and notice that "straight uphill" is 20° west of north, and that the slope in that direction is $.18$.

1. Find the gradient at your location.
2. Find the slope if you were to walk in the direction $[3; 4]$.
3. In which directions is the slope zero?
4. Draw a line splitting the plane into halves: uphill and downhill directions.

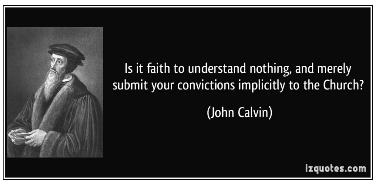
Concavity

Let $\nabla f = [f_x; f_y]$ and $H = \begin{bmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{bmatrix}$ be evaluated at a given point. Let $\vec{v} = [v_1; v_2]$ be velocity.

- slope
  $D_{\vec{v}} f = \frac{1}{\|\vec{v}\|} \nabla f \cdot \vec{v} = \frac{1}{\|\vec{v}\|}(v_1 f_x + v_2 f_y)$
- concavity
  $D_{\vec{v}}(D_{\vec{v}} f) = \frac{1}{\|v\|^2} \begin{bmatrix} v_1 f_{xx} + v_2 f_{xy} \\ v_1 f_{yx} + v_2 f_{yy} \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$
  $= \frac{1}{v \cdot v}(v_1^2 f_{xx} + 2 v_1 v_2 f_{xy} + v_2^2 f_{yy}) = \frac{v^T H v}{v^T v}$

Do a numerical example.

Implicit Functions



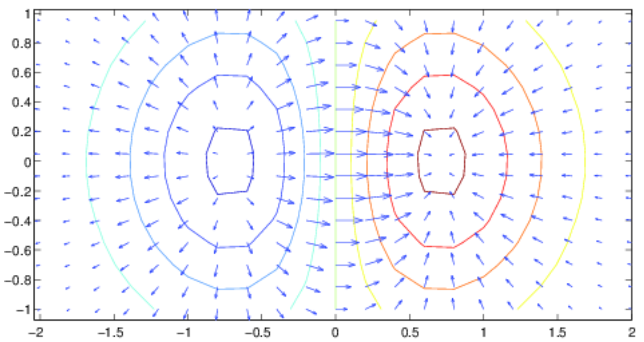
When it's inconvenient to solve for a "dependent" variable, the graph defines a function implicitly.
Sometimes expressed as solution set of $F(x, y, z) = \text{constant}$.

Implicit Differentiation

Suppose z is an implicit function of x and y. Then to find $\frac{\partial z}{\partial x}$, treat y as constant, but z as a function of x. Differentiate both sides w.r.t. x, and solve for $\frac{\partial z}{\partial x}$.

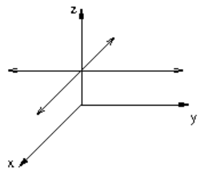
1. $x^2 + y^2 + z^2 = 121$
   find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(2, 6, 9)$.
2. If $z^3 x = \ln(xyz)$, then find z_x and z_y.
3. find $\frac{\partial z}{\partial x}$ if $x(y^2 z + e^{-z^2}) = 1$

Gradient Field



Continuity

$$f(x, y) = \begin{cases} 1 & xy = 0 \\ 0 & \text{otherwise} \end{cases}$$



1. Find ∇f.
2. Find $\lim\limits_{x,y \to 0,0} f(x, y)$

Black-Scholes

Call option price $C(s,t)$. Today $s = 540$.

- $C = 14.82$
- $\frac{\partial C}{\partial s} = 0.683$ (delta)
- $\frac{\partial C}{\partial t} = -0.591$ (theta)

Estimate C tomorrow if s increases to 543.

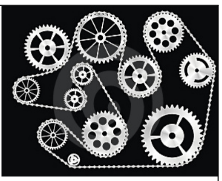
Intermediate Variables

Let $z = f(x,y) = x^2 y$, with $x = t^3$ and $y = t^4$.

1. Substitute and find $\frac{dz}{dt}$.
2. Find ∇z, $x'(t)$, and $y'(t)$.
3. Check that $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

There is one term for each intermediate variable. The derivative $\frac{dz}{dt}$ is the rate of change in z with respect to t, as x and y move parametrically.

Chain Rule



Theorem 6

Suppose y is a function of $\vec{u}$, and $\vec{u}$ is a function of $\vec{x}$. Then

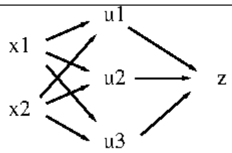
$$\frac{dy}{d\vec{x}} = \frac{dy}{d\vec{u}}\frac{d\vec{u}}{d\vec{x}}$$

Chain Rule Pattern

A derivative is the rate of change of one variable with respect to another.

- Sketch input $\longrightarrow$ output layer network.
- Multiply derivatives that link the input to the output.
- Add terms, each one corresponding to a possible path of dependency.

Numerical Example



Given $\frac{dz}{d\vec{u}} = [5, 2, 4]$ and $\frac{d\vec{u}}{d\vec{x}} = \begin{bmatrix} 3 & -2 \\ 6 & 7 \\ -1 & 8 \end{bmatrix}$

1. $\frac{dz}{dx_1} = (5)(3) + (2)(6) + (4)(-1) = 23$
   Note this is a dot product.
2. All else equal, what happens to z if you turn the x_1 knob by $.03$?
   $\Delta z \approx dz = \frac{dz}{dx_1}\Delta x_1 + \frac{dz}{dx_2}\Delta x_2$

Formula Example

$z = (u + 2v)^3$, $u = x^2 y$, $v = e^x y$

$$\frac{dz}{d\vec{x}} = [3(u+2v)^2,\ 6(u+2v)^2] \begin{bmatrix} 2xy & x^2 \\ e^x y & e^x \end{bmatrix}$$

Expanded out,

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = 6y(u+2v)^2(x + e^x)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = 3(u+2v)^2(x^2 + 2e^x)$$

Cash Flow

- Pops gives $5\%, 7\%, 10\%$ of his income to Dad, Uncle, Aunt respectively.
- Dad gives 15% to you.
- Uncle gives 4% to you.
- Aunt gives 6% to you.

If Pops wins \$1000, how much will you (Y) get?
Note the proportionality: $dY = \frac{dY}{dP} dP$

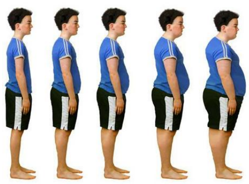
Voltage

$V = IR$ (voltage, current, resistance).
Find $\frac{dI}{dt}$ when

- $R = 600$ ohms, $I = .04$ amps
- $\frac{dR}{dt} = .5$ (heating up)
- $\frac{dV}{dt} = -.01$ (battery draining)

$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}$

BMI



A person that weighs w kg
and is h cm tall
has a body mass index of

$$B = w(h/100)^{-2}$$

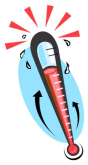
A boy currently 140 cm tall weighs 33 kg. He is growing at 0.6 cm/month and 0.4 kg/month. Find the rate of change in his BMI.

From Polar

Let $f(x, y) = \frac{y}{x^2+1}$. Using the polar change of variables, find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

- $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{-2xy\cos\theta}{(x^2+1)^2} + \frac{\sin\theta}{x^2+1}$
- $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{2xyr\sin\theta}{(x^2+1)^2} + \frac{r\cos\theta}{x^2+1}$

Temperature



Suppose temperature is given by

$$T(x, y) = \exp\left(\frac{12xy - x^2 - y^4}{50}\right)$$

degrees K. Find the extreme temperature(s).

- At an extreme, no improvement in any direction.
- $\arg\max x, y\, T(x, y) = \arg\max x, y(12xy - x^2 -$

Extrema

Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^n$.

Definition 7

$f(\vec{x}_0)$ is a local maximum if $\exists \epsilon > 0$ such that $f(\vec{x}) \leq f(\vec{x}_0)$ whenever $\|\vec{x} - \vec{x}_0\| < \epsilon$

$f(\vec{x}_0)$ is a global (absolute) maximum if $f(\vec{x}) \leq f(\vec{x}_0)$ for all $\vec{x} \in D$

- local/global minimum defined similarly
- maximums and minimums are collectively called extrema

Searching for Extrema

Definition 8

The point $\vec{x}_0$ is a critical point if $\nabla f(\vec{x}_0) = \vec{0}$, or if the gradient is not defined.

Theorem 9 (Fermat's Theorem)

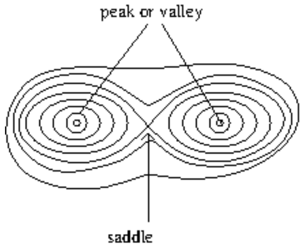
Extrema can exist only at critical points or on the boundry of the domain D.

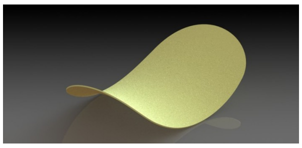
Saddle Points

Definition 10

A saddle point is a critical pt having both higher and lower function values arbitrarily close by.

Contours look like concentric ellipses at extrema, and hyperbolas at saddles.





2nd Derivative Test

Theorem 11 (2nd Derivative Test)

Suppose $\nabla f(a, b) = \vec{0}$, and f is locally smooth enough. Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

- $D > 0$, $f_{xx} > 0$ *implies* $f(a, b)$ *is local min.*
- $D > 0$, $f_{xx} < 0$ *implies* $f(a, b)$ *is local max.*
- $D < 0$ *implies* $f(a, b)$ *is a saddle.*
- *If* $D = 0$, *the test is inconclusive.*

Local Concavity

Justify the 2nd deriv. test:

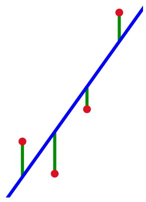
- slope $D_{\vec{v}}f \propto f_x v_1 + f_y v_2$ (equals 0 for all $\vec{v}$ iff $\nabla f = \vec{0}$)
- concavity $D_{\vec{v}}(D_{\vec{v}}f) \propto (f_{xx}v_1 + f_{yx}v_2)v_1 + (f_{xy}v_1 + f_{yy}v_2)v_2 = f_{xx}v_1^2 + 2f_{xy}v_1v_2 + f_{yy}v_2^2$
- $v_2 = 0$: then concavity $\propto f_{xx}$
- $v_2 \neq 0$: WLOG scale $\vec{v}$ so that $v_2 = 1$, then concavity $\propto f_{xx}v_1^2 + 2f_{xy}v_1 + f_{yy}$
- by QF, stays same sign as long as $f_{xx}f_{yy} - f_{xy}^2 > 0$

Examples

Use the gradient and Hessian to find and classify all critical points of the objective function.

1. $z = 4 + x^3 + y^3 - 3xy$
2. $z = \frac{x^3}{3} + \frac{4y^3}{3} - x^2 - 3x - 4y - 3$ (visualize in Maxima)
3. mvopt-apps.pdf #5
4. Find shortest distance from point to plane.

Best Fit Line



Find the line $y = a + bx$ that has least squares error in fitting the points:

$$\{(0, 15), (3, 24), (5, 32), (9, 50)\}$$

The objective function is $f(a, b) = (a - 15)^2 + (a + 3b - 25)^2 + (a + 5b - 33)^2 + (a + 9b - 48)^2$.

General formula for a and b given $\{(x_i, y_i)_{i=1}^n\}$?

Not In Kansas Anymore

Find and classify the critical points of

$$f(x, y) = (x^2 - 1)^2 + (x^2 - e^y)^2$$

Notice anything weird?

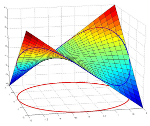
Global/Absolute Extrema

- The extreme value theorem guarantees that a continuous function attains its minimum and maximum on a closed and bounded domain.
- Candiate locations for global extrema are critical points and boundary points.
- On the boundary, substitute and find 1 variable critical pts; check corners.
- Evaluate the objective function at each candidate and select the highest and lowest.

Examples

1. A flat circular plate covers $x^2 + y^2 \leq 1$. The temperature at a given point on the plate is $T(x, y) = x^2 + 3y^2 - x$. Find the hottest and coldest points on the plate.
2. Let $f(x, y) = 2x^2 - 4x + y^2 - 4y + 1$ be defined on the triangle bounded by $x = 0$, $y = 3$, and $y = x$. List all points where an absolute extremum may occur, and evaluate f at each one.

Constrained Optimization

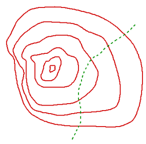


Optimize objective function $f(x,y) = x^2 + 2y^2$ subject to (s.t.) the constraint $y = x^2 - 4$.

- Draw the contours of f, along with the constraint path.
- substitution method: plug $x^2 = y + 4$ into f and make it a calc I problem in $y \in [-4, \infty)$
- note that you are changing elevation as you cross contours, so at extrema the contours must be parallel to the path.

Lagrange Optimization

- Constraint is a contour of the surface $g(x,y) = y - x^2 + 4$.
- At extrema, the gradient directions of f and g must coincide.
- Lagrange system of equations:
  $\nabla f = \lambda \nabla g$
  $g(x,y) = 0$



- λ is the Lagrange multiplier
- solve eqns and evaluate objective func. to find potential extrema (Maxima)

Milkmaid Problem

The problem is described [here]. In particular, suppose

- The maid is at $(-1,0)$.
- The cow is at $(1,0)$.
- The objective function is $f(x,y) = \sqrt{(x+1)^2 + y^2} + \sqrt{(x-1)^2 + y^2}$, which has elliptical contours.
- The river's course is described by $(x^2 + 3y - 6)(y - 2) = 3x$.

Example

Find the point on the curve $x^2 + xy = 1$ closest to the origin.

$\min x^2 + y^2$
s.t. $x^2 + xy - 1 = 0$

Show that $(.841, .348)$ satisfies the Lagrange equations.

Example

Find the volume of the largest rectangular box, having sides parallel to coordinate planes, and inscribed in the ellipsoid $16x^2 + 4y^2 + 9z^2 = 144$.

- label one corner (x, y, z), so the objective function is $V(x, y, z) = 8xyz$
- the constraint is $g(x, y, z) = 16x^2 + 4y^2 + 9z^2 - 144$
- Lagrange system of equations:
  $8yz = \lambda(32x)$
  $8xz = \lambda(8y)$
  $8xy = \lambda(18z)$
  $16x^2 + 4y^2 + 9z^2 - 144 = 0$

Optimization Summary

The objective function f expresses the quantity you want maximized or minimized. Identify independent variables. Note the feasible region, including potential constraint $g = 0$.

- Unbounded domain: local extrema occur at critical pts; classify using 2nd D. test
- Bounded domain: also check boundary and corners; evaluate f to select global extrema
- Constrained: solve Lagrange equations for potential extrema

Example

Optimize $z = x^3 + y^3 + (x-2)^2 + (y-5)^2$ on the region bdb $y = 0$ and $y = 4 - x^2$.

1. Use Maxima to graph on $[-4, 4] \times [4, 4]$.
2. Find and classify interior critical points.
3. Solve the Lagrange eqns in Maxima for parabolic boundary.
4. Find the absolute extrema.