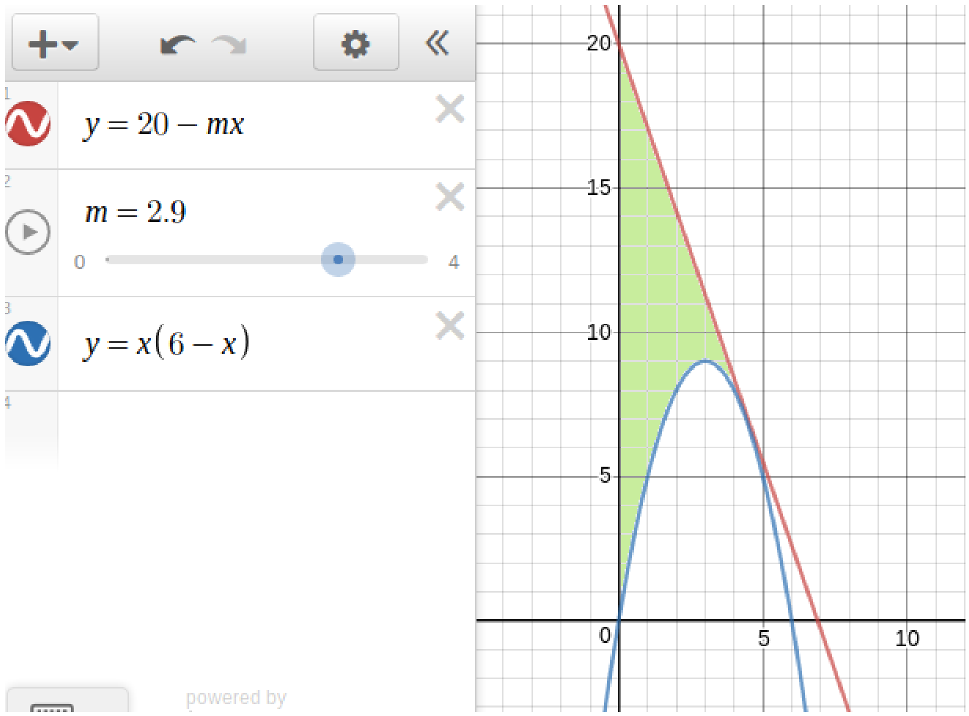
Warm-Up

1. A rotating straight arm swings clockwise from a pivot $(0, 20)$ until it hits an obstacle occupying the region above the x-axis and below $y = x(6 - x)$.



   (a) For the shaded region, find the
      i. exact slope of the line
      ii. location of the cusp (point of tangency)
      iii. angles (in degrees) at the other corners
      iv. perimeter
      v. area

   (b) Generalize by letting the pivot be $(0, b)$ and the curve be $y = ax(6 - x)$.
      i. Use Desmos to estimate the derivative of area with respect to the parameters a and b, when tuned to $a = 1$ and $b = 20$. (compute difference quotients).
      ii. Write area, $A(a, b)$ as a "multivariable" function of both a and b.
      iii. For a given value of b, what values of a allow the intersection to occur in the 1st quadrant ?

2. $f(x) = x^2$ has average height of 100 over the interval $[5, b]$. Find b “by hand”.

Polar Coordinates

3. Convert $\left(\frac{5\pi}{12}, 20\right)_P$ to Cartesian coordinates.
4. Convert $(-4, 7)_C$ to polar coordinates.
5. Convert $(\frac{\pi}{2}, -6)_C$ to polar coordinates, and use degrees to measure the angle.
6. List 4 aliases of $(\frac{\pi}{4}, 2)_P$.
7. How can you describe all aliases of $(\frac{\pi}{3}, 7)_P$? (Hint: use expressions involving an integer k)
8. Sketch the region described by
   (a) $r \in (2, 3]$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.
   (b) $|r| \leq 3$ and $|\theta| \leq \frac{\pi}{4}$
9. Find a Cartesian equation and describe the graph.
   (a) $\theta = \frac{\pi}{3}$
   (b) $r = 2\cos\theta$
   (c) $r = \sec(\theta)\tan(\theta)$
10. Find a polar equation and describe the graph.
    (a) $x^2 + y^2 = 25$
    (b) $y = 1$
    (c) $x + 2y = 10$
11. Show that $r = a\sin\theta + b\cos\theta$ is a circle. Find the center and radius.
12. The curve $r = \sin(\theta)\tan(\theta)$ is called a "cissoid". Prove that the graph lies inside the vertical strip $x \in [0, 1)$.
13. Use Desmos to investigate the family of curves
    $$r = \frac{1 - a\cos(\theta)}{1 + a\cos(\theta)}$$
    Find transitional values of a where the curve changes its basic shape.
14. Sketch the polar curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$ on the same graph, and determine where they intersect in Cartesian coordinates.
15. Sketch $r = 1$ and $r = 2\cos\theta$ on the same graph. Label the intersections. Shade the region bounded between the curves.
16. Consider $r = \exp(\theta)$ for $\theta \in [0, 2\pi]$.
    (a) Sketch the graph.
    (b) Find points where the graph is horizontal.
    (c) Find points where the graph is vertical.
17. Let $r = \sin(2\theta)$.
    (a) Describe the shape of the graph.
    (b) Find the expression for $\frac{dy}{dx}$ in terms of θ (don't simplify).
    (c) Find the slope of the graph when $\theta = \frac{\pi}{6}$.
    (d) Find the equation of the tangent line at that point.

18. Suppose $r(\frac{\pi}{8}) = 17$ and $\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{8}} = 2.68$. Use differentials to estimate $x(\frac{\pi}{9})$ and $y(\frac{\pi}{9})$.

Parametric Equations

19. Write using equations parameterized by t, and sketch the graph.

    (a) $y = e^{-x}$, $x \in [-2, 2]$

    (b) $r = e^{-\theta}$, $\theta \geq 0$

20. Parameterize a circle with area 154 that is centered at $(0, 12)$. For $t \in [0, \infty)$ minutes, the position should start at the top of the circle and move counter-clockwise with a frequency of 20 RPM.

21. Parameterize this ellipse for $t \geq 0$ seconds.

$$\left(\frac{x-6}{4}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

starting at $(2, 2)$, moving clockwise, with a period of 24 seconds.

22. Plot points to sketch parametric equations $(e^{2t}, t+1)$ for $t \in \mathbb{R}$. Then eliminate the parameter to write y as a function of x, and state the domain.

23. Do this matching problem from the Stewart textbook:

28. Match the parametric equations with the graphs labeled I-VI. Give reasons for your choices. (Do not use a graphing device.)

(a) $x = t^4 - t + 1, \quad y = t^2$

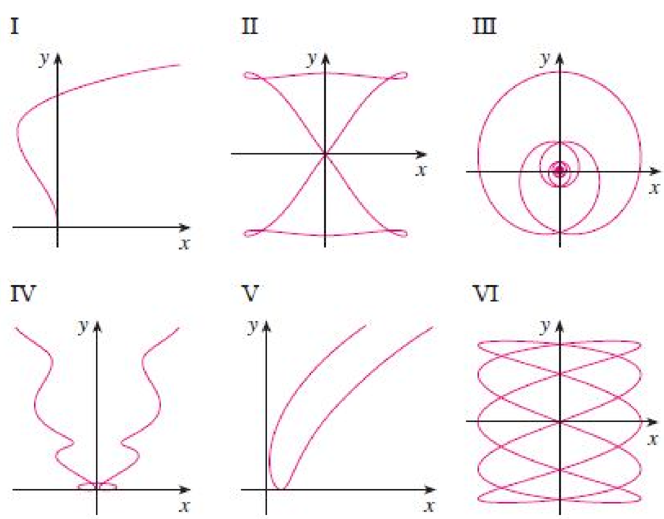
(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

(c) $x = \sin 2t, \quad y = \sin(t + \sin 2t)$

(d) $x = \cos 5t, \quad y = \sin 2t$

(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$

(f) $x = \dfrac{\sin 2t}{4 + t^2}, \quad y = \dfrac{\cos 2t}{4 + t^2}$



24. Suppose $(x(t), y(t))$ are parametric equations that describe motion that covers a path for $t \in [0, 12]$. Modify the equations to:

   (a) Shift the path 4 units north, and 7 units west.

   (b) Shift the path 9 units at an angle of θ radians counter-clockwise from east.

   (c) Move twice as fast to cover the path for $t \in [0, 6]$.

   (d) Move backwards to cover the path for $t \in [0, 4]$.

   (e) Double the size of the path, while maintaining its shape and starting point.

   (f) Bonus: rotate the path θ radians counter-clockwise around the origin.

25. Use Desmos to investigate the "Lissajous figures" described by $x = a\sin(nt)$ and $y = b\cos(t)$ for $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$.

26. Find the tangent line to the graph described parametrically by $\left(\frac{20}{1+t^2}, 5(.8)^t\right)$ when $t = 2$.

27. Let $y = \frac{1}{1+x}$ for $x \geq 0$. The obvious parameterization is $\left(t, \frac{1}{1+t}\right)$ for $t \geq 0$. Re-parameterize in such a way that the same path is traced for $t \in [0, 1)$.
Hint: let x be a one-to-one function with domain $[0, 1)$ and range $[0, \infty)$

Length and Area

28. Find the length of these curves:
    (a) $y = xe^{-x}$ for $x \in [0, 2]$
    (b) $r = \sin(2\theta)$ for $\theta \in [0, \frac{\pi}{2}]$
    (c) $(\sin(t), e^{-t^2})$ for $t \in [-\pi, \pi]$
29. Find the area of the region bounded by:
    (a) $r = \sin(2\theta)$ for $\theta \in [0, \frac{\pi}{2}]$
    (b) $(\sin(t), e^{-t^2})$ for $t \in [-\pi, \pi]$
30. Find the perimeter and area of the region bounded by x-axis and the parametric curve:

$$x = (t-1)^2$$

$$y = (t-2)(t-5)$$

31. An elliptical clock has period 12, starts at $(0, 2)$, and is at $(3, 1)$ when $t = 3$.
    (a) Parametrize the path.
    (b) Find the equation of the tangent line at $t = 2$.
    (c) Set up the integral to find the distance traveled for $t \in [0, 2]$.
    (d) At what time has it traveled 5 units?
    (e) Find the maximum and minimum speeds.
    (f) Find the area of the region bounded by the ellipse and $|x| = 2$.
32. Find a general formula for the speed of an object moving on the ellipse:

$$x = a\cos(\omega t) \qquad\qquad y = b\sin(\omega t)$$

Simplify the formula so that it uses only one trig function.

33. Consider the Carson-Newman seal pictured here:

Suppose the boundary of the seal ranges between 62 and 68 units from the center, following a wave pattern.

(a) Find constants b, c, k to write the boundary in the form $r(\theta) = b + c\cos(k\theta)$ for $\theta \in [0, 2\pi]$.

(b) Graph with Desmos. For fun, see what happens to the graph if k is not an integer, and you let θ go past 2π.

(c) Find the perimeter. (write the integral, and evaluate with software)

(d) Find the area. (write the integral, and evaluate with software)

(e) Find the "average" radius, i.e. the radius of a circle that has the same area. Does it equal b ?

(f) Warp the seal by increasing c until the "average" radius is 70. (Hint: create a slider in Desmos)

(g) Now evaluate the area by hand, using general non-zero constants b, c, and k. Show that the area simplifies to $A = \pi(b^2 + \frac{c^2}{2})$ if $k \geq 1$ is an integer. (you may need a double-angle formula). Note that area is the same for any positive integer k.

Vectors, Space, Lines, Planes, Spheres

34. Find a vector with length 20, that points in the opposite direction of $\vec{v} = [5, -3, 9]$.

35. Write the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ associated with the polar point $(5, \frac{\pi}{8})_P$ in $\mathbb{R}^2$.

36. Consider points $P(1, 8, -2)$ to $Q(3, 2, 7)$ in $\mathbb{R}^3$.
    (a) find $\text{dist}(P, Q)$
    (b) write a vector that points from P to Q
    (c) write a vector that points from Q to P
    (d) write a vector that points from P through Q, and then another 1 unit beyond
    (e) write a vector that points from the origin to the midpoint of $\vec{PQ}$
    (f) find the equation of the sphere centered at P with volume 100 cubic units
    (g) find the equation of the sphere with $\vec{PQ}$ as a diameter

37. Find two unit vectors in $\mathbb{R}^2$ parallel to the tangent line of $y = e^{-x^2}$ at $x = 1$.

38. True or false? Justify your answer.
$$\|\vec{a} - \vec{b}\| = \|\vec{a}\| - \|\vec{b}\|$$

39. This problem is from a physics book.

    85 You are kidnapped by political-science majors (who are upset because you told them political science is not a real science). Although blindfolded, you can tell the speed of their car (by the whine of the engine), the time of travel (by mentally counting off seconds), and the direction of travel (by turns along the rectangular street system). From these clues, you know that you are taken along the following course: 50 km/h for 2.0 min, turn 90° to the right, 20 km/h for 4.0 min, turn 90° to the right, 20 km/h for 60 s, turn 90° to the left, 50 km/h for 60 s, turn 90° to the right, 20 km/h for 2.0 min, turn 90° to the left, 50 km/h for 30 s. At that point, (a) how far are you from your starting point, and (b) in what direction relative to your initial direction of travel are you?

40. Consider these spheres:
    - $(x-3)^2 + y^2 + z^2 = 25$
    - $x^2 + y^2 + z^2 = 24x - 10y + 16z - 229$

    (a) Find the center and radius of each sphere.
    (b) Find the distance between the spheres.
    (c) Find an equation describing the set of all points equidistant from the center of each sphere.
    (d) Plot that equation from part (c), along with both spheres, in Geogebra3D.

41. Three points are **co-linear** if they lie in the same line. Let $P(4,2,7)$, Q, and R be co-linear. Find coordinates for Q and R if $\vec{PQ} = [-2,5,3]$ and $\vec{QR} = -1.5\vec{PR}$.

42. Let $(x-6)^2 + (y-1)^2 + (z-5)^2 = 75$ be a sphere.

   (a) Is the point $P(9,-4,11)$ inside, outside, or on the sphere ?

   (b) Find two points where the sphere intersects the x-axis.

   (c) Find the distance from $Q(2,10,8)$ to the sphere.

43. There is a sphere S_1 with equation $(x-5)^2 + (y-7)^2 + (z-4)^2 = 9$. Find the equation of another sphere S_2 with:

   - 75% more volume than S_1
   - center co-linear with the origin and S_1's center
   - 10 units distance between S_1 and S_2

44. Find the angle (in degrees) between vectors $[1,6,4]$ and $[2,5,1]$.

45. Find c so that $[1,c,3]$ and $[-2,2,4]$ are orthogonal.

46. Let $\theta(a)$ be the angle (in degrees) between the vectors $[2,a,7]$ and $[a^2,1,5]$.

   (a) Find $\theta(3)$.

   (b) Find $\left.\frac{d\theta}{da}\right|_{a=3}$.

47. Find the equation of the plane through $(5,8,3)$ and orthogonal to the vector $[7,-2,4]$.

48. If $\|\vec{u}\| = 7$, $\|\vec{v}\| = 12$, and the sine of the angle betweeen them is 0.75, then find two possible values for $\vec{u}\cdot\vec{v}$. What is the smallest possible value of $\|\vec{u}-\vec{v}\|$?

49. Find values of a such that the angle between $[2,1,-1]$ and $[1,a,0]$ is $45°$.

50. Find the acute angle between $y = x^2$ and $y = x^3$ where they intersect at $(1,1)$.

51. $\|\vec{PQ}\| = 19$, $\|\vec{RQ}\| = 26$, and $\vec{PQ}\cdot\vec{RQ} = 494$. Explain how you know P,Q,R are co-linear. Find $\text{dist}(P,R)$.

52. Find the acute angle (in degrees) between $\vec{v} = [1;1;1]$ and

   (a) the y-axis

   (b) the xy-plane

53. Find the unit vectors in $\mathbb{R}^2$ that make a $20°$ angle with the vector $[1,7]$.

54. Show that if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.

55. A sphere is centered at $(7, -2, 15)$ and the vector $\vec{v} = [9, 13, 5]$ goes from the center to the surface of the sphere.

    (a) Find the equation of the sphere.

    (b) Find the equation of the axial line through the sphere's center in the direction of $\vec{v}$.

    (c) Find the equation of the plane, normal to $\vec{v}$, that splits the sphere into two hemispheres.

56. Let $P(2, 4, 9)$, $Q(7, 3, 5)$, and $R(1, 8, 2)$ be three points in $\mathbb{R}^3$.

    (a) Find the equation of the plane containing those points.

    (b) Find the equation of the line through P that is perpendicular to the plane.

    (c) Find the area of $\triangle PQR$.

57. Consider the line $\vec{\ell}(t) = \begin{bmatrix} 3+5t \\ 1-7t \\ 8+2t \end{bmatrix}$ and the plane $z - 15 = 2(x-3) + 9(y+1)$.

    (a) Find the point of intersection.

    (b) Find the acute angle (in degrees) of intersection.

58. Consider the line $\vec{\ell}(t) = \begin{bmatrix} 3+5t \\ 1-7t \\ 8+2t \end{bmatrix}$ and the sphere $(x-3)^2 + (y+1)^2 + (z-15)^2 = r^2$.

    (a) Find the values of t, and the correspondings point of intersection if $r = 10$.

    (b) Find the EXACT value of r such that line is tangent to the sphere (one intersection). Then find that EXACT point of tangency.

    (c) Using a computer, find the value of r such that the points of intersection are 50 units apart.

59. These two lines are known to intersect. What is the value of c ? Follow these steps:

$$\vec{\ell}_1(t) = \begin{bmatrix} 4t \\ 2t-7 \\ 3t-2 \end{bmatrix} \qquad \vec{\ell}_2(t) = \begin{bmatrix} 14-6t \\ 12+t \\ 5+ct \end{bmatrix}$$

    (a) Using different symbols for the parameters, e.g. t_1 and t_2, set the x and y coordinates equal and find the point of intersection. (you are ignoring the z-coordinates for now)

    (b) Now set the z coordinates equal to find the value of c.

60. Find the line of intersection between these planes. Follow the steps:

$$z + 80 = 2(x-3) + 5(y+1) \qquad 4x + y + 3z = 45$$

    (a) Set $x = 0$ and solve for y and z to find one point on the line.

    (b) Set $z = 0$ and solve for x and y to find one point on the line.

    (c) Write parametric equations of the line through those two points.

61. Find the value of z so that the point $(8, 2, z)$ is on the plane $3(x - 8) + 5(y - 2) + 9(z - 1) = 36$.

62. Find an equation for the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$.

63. Consider points $P(1, 3, 5)$, $Q(2, 4, 8)$, $R(4, 9, 11)$, and $S(x, 7, 2)$.

    (a) Let $x = 6$ and find the volume of the parallelpiped with corners at P, Q, R, S.

    (b) Find the value of x so that S is co-planar with ΔPQR.

64. A parallelpiped determined by points: $P(0, 0, 0)$, $Q(3, 1, 2)$, $R(-1, 4, 1)$, $S(2, 5, z)$, has volume 100.

    (a) If $z > 0$, find its value.

    (b) If $z < 0$, find its value.

Multi-variable Functions

65. Consider a multivariable function with $f(29, 52) = 73$ and $\nabla f(29, 52) = \begin{bmatrix} -0.23 \\ 0.09 \end{bmatrix}$.

    (a) Find the equation of the tangent plane at the given point.

    (b) Estimate $f(29.5, 50.2)$

    (c) Find the equation of the normal line.

    (d) Let $\vec{v} = [a; 7]$. If $a = -2$, find $D_{\vec{v}} f(29, 52)$.

    (e) Find the value of a such that $D_{\vec{v}} f(29, 52) = 0$

66. Let $f(x, y) = \frac{5x}{1+y^2}$.

    (a) Evaluate $f(9, 3)$.

    (b) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as functions of x and y.

    (c) Evaluate the gradient $\nabla f(9, 3)$.

    (d) Find the equation of the tangent plane at $(9, 3)$.

    (e) Find the equation of the normal line.

    (f) Using the linearization (i.e. tangent plane) estimate $f(9.17, 2.88)$.

    (g) Evaluate $f(9.17, 2.88)$ exactly using the non-linear formula for f.

    (h) Find the directional derivatives $D_{\vec{v}} f(9, 3)$ for these directions:

        i. $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

        ii. $\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

        iii. $\vec{v} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

        iv. $\vec{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$

        v. $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$

        vi. $\vec{v} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

        vii. $\vec{v} = \begin{bmatrix} -5 \\ -12 \end{bmatrix}$

        viii. $\vec{v} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

67. Let $z = x^2 - y$. The contour that contains (3,2) also passes through $(8, y)$ for what value of y?

68. Let $z = f(x, y) = \exp(-x^2 - y^2)$.

    (a) Find the range.

    (b) Find the radius of the circular contours for $z = .5$, $z = .2$, and $z = .1$.

69. Let $z = f(x, y) = \log_2(xy + 1)$.

    (a) Find the domain and shade it on the xy-plane.

    (b) Sketch the contours corresponding to $z = -2$, $z = -1$, $z = 0$, $z = 1$, and $z = 2$.

70. Contrast the contours of $z = 2x + y$ and $z = \sin(2x + y)$.

71. Let $f(x, y) = |x| + |y|$. For this problem, consider contours for $z = 0, 1, 2, 3, 4, 5$

    (a) Sketch the contours of $z = f(x, y)$. Describe the shape and spacing.

    (b) Sketch the contours of $z = f^2(x, y) = (|x| + |y|)^2$. Describe the shape and spacing.

    (c) Sketch the contours of $z = \sqrt{f(x, y)} = \sqrt{|x| + |y|}$. Describe the shape and spacing.

72. Here is a table of points for a function $z = f(x, y)$.

x	y	z
7.00	3.85	14.50
7.03	4.00	14.90
6.97	4.00	15.40
7.00	4.15	16.10

    (a) Average the surrounding points to estimate $f(7, 4)$.

    (b) Use difference quotients to estimate $\nabla f(7, 4)$.

    (c) Use those answers to write an approximate tangent plane at $(7, 4)$.

    (d) Estimate $f(7.05, 4.09)$.

73. Suppose at a certain point, $\nabla f = \begin{bmatrix} 0.21 \\ -0.56 \end{bmatrix}$ and the Hessian is $H = \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$. Consider the velocity direction $\vec{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

    (a) Find the slope: $D_{\vec{v}} f$.

    (b) Find the concavity: $D_{\vec{v}}(D_{\vec{v}} f)$.

    (c) If you move in the opposite direction, $-\vec{v}$, then what are the slope and concavity ?

    (d) Find the steepest ascent direction, and the slope in that direction.

    (e) Find the steepest descent direction, and the slope in that direction.

    (f) Find a direction where the slope is zero.

    (g) Find a direction where the slope is 0.35.

    (h) Find a direction where the slope is -0.35.

74. Let $f(x, y) = x^3 + \frac{15x}{y}$

    (a) Find the gradient $\nabla f(x, y)$; then evaluate $\nabla f(2, 5)$.

    (b) Find the Hessian matrix $H(x, y)$; then evaluate $H(2, 5)$.

(c) Find the equation of the tangent plane at $(2, 5)$.

(d) Find the equation of the normal line at $(2, 5)$.

75. This plane lies tangent to the surface of $f(x, y)$.

$$z = 425 + 3(x - 90) + 7(y + 260)$$

(a) Find the evident point P of tangency.

(b) Find ∇f at P.

(c) Find the slope of the surface in the steepest ascent direction.

(d) Estimate $f(86, -257)$.

(e) Find the equation of the line through P orthogonal to the surface of f.

(f) Find the equation of the tangent line to the contour $f(x, y) = 425$.

76. If $dz = 2dx + 5dy + 4dt$, find the equation of the tangent plane at $(7, 0, 3, 14)$.

77. You have a company that sells snowshovels. Quantity sold (S) is a function of price (P dollars), temperature (T degrees), and advertising (A units). Suppose $dS = -25dP - 7dT + 4dA$.

(a) Find $\frac{\partial S}{\partial P}$, and write a sentence that explains its sign.

(b) Find $\frac{\partial S}{\partial T}$, and write a sentence that explains its sign.

(c) Find $\frac{\partial S}{\partial A}$, and write a sentence that explains its sign.

(d) If temperature falls 10 degrees, you raise the price by a dollar, and reduce advertising by 6 units, estimate the net affect on S.

78. A bird is flying south for the winter. Let $T(t, x, y)$ be the expected daily high temperature as a function of time in days, degrees east longitude, and degrees north latitude. At the current time and location, suppose $T = 62$ and $\nabla T = [-0.18, .07, -1.53]$. Over the next week, the bird hopes to move $2°$ east and $5°$ south. Find the differential dT.

79. Suppose at a certain point, the surface of a function has

- slope 7 in the direction $[3, 4]$
- slope 1 in the direction $[8, 15]$

Find ∇f.

80. Suppose $f(200, 700) = 125$, $f(201, 704) = 143$, and $f(198, 703) = 119$.

(a) Estimate $\nabla f(200, 700)$.

Hint: $f(200 + dx, 700 + dy) - f(200, 700) \approx \nabla f(200, 700) \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$

(b) Estimate $f(203, 695)$.

81. Suppose you are on a hill and the steepest descent direction is east by southeast (i.e. $\theta = -11.25°$), having slope -0.35. Find the slope in the southwest direction ($\theta = 225°$).

82. Consider the surface of $f(x, y) = xy$ and the line $\ell(t) = \begin{bmatrix} 1 + 2t \\ 2 + 3t \\ 146 - 23t \end{bmatrix}$.

(a) Find the points of intersection.

(b) Find the acute angles (in degrees) at which the line intersects the surface at those points.

83. Let $f(x,y) = \frac{x^2}{y}$.

   (a) Find the equation of the tangent plane at $(6,4)$.
   (b) Find the slope of the surface in the direction of $\vec{v} = [8, -5]$.
      i. At an arbitrary point (x,y).
      ii. At $(6,4)$.
   (c) Find the concavity in the direction of $\vec{v}$.
      i. At an arbitrary point (x,y).
      ii. At $(6,4)$.

84. Find all points at which the direction of fastest change of $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

85. Show that $v = (x - at)^4 + \cos(x + at)$ satisfies the wave equation:

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$$

86. Show that $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ satisfies the Laplace equation:

$$f_{xx} + f_{yy} = 0$$

87. Suppose you manage a fleet of trucks that get highway fuel economy modeled by:

$$y = \frac{x}{2 + (.02x)^4}$$

   where x is speed in MPH, and y is your gas mileage in MPG.

   (a) Plot y as a function of x, and use calculus to find the speed that maximizes gas mileage.
   (b) If gas is \$2.00 per gallon, and drivers are paid \$15 per hour, how fast should they drive to minimize total monetary costs? What would be the fuel economy? What is the total cost per mile? Give your answers in the form of a paragraph.
   (c) Suppose gas is G dollars per gallon, and drivers are paid T dollars per hour. Let X_o be the optimal speed to minimize total costs. Write X_o as a function of G and T.
   (d) Plot the contours $X_o(G,T) = \{55, 65, 75\}$.
   (e) If gas is \$2 per gallon, what value does a 65 MPH speed limit put on the drivers' time?
   (f) Find $\frac{\partial X_o}{\partial G}(2,15)$ and $\frac{\partial X_o}{\partial T}(2,15)$.
   (g) Write the linear approximation (tangent plane) to X_o at $G = 2$ and $T = 15$.
   (h) Use only $\nabla X_o(2,15)$ to answer this problem. If gas goes up 10 cents, then approximately what increase in driver pay would leave the optimal speed unchanged?
   (i) Suppose gas goes up 25 cents and the drivers get a \$5 raise. (So now $G = 2.25$ and $T = 20$).
      i. You should instruct the drivers to go _______ MPH (faster | slower).
      ii. Use the linear approximation to answer the previous question, and contrast the approximate and exact answers.
      iii. Using the exact optimal speed, find the new cost per mile.
      iv. If your your drivers log about 300 thousand miles per year, then the increase in gas and labor prices will increase your total expenses by _______ thousand dollars.

88. Let $f(x, y) = \ln(10 + x^2 + 3x + y^2)$
   (a) Find $\nabla f(x, y)$, then evaluate $\nabla f(2, 5)$.
   (b) Find the Hessian matrix $H(x, y)$.
   (c) Let $(2, 5)$ be your current location, and consider walking in the walking in the 11:20 clock direction.
      i. Write a unit vector representing your direction (use cos/sin of your angle).
      ii. Find the slope.
      iii. Find the concavity using the formula $\vec{v} \cdot (H\vec{v})$.

Chain Rule

89. Let z be a function of x and y, and let both x and y be functions of s and t. At a certain point,
$$\nabla z = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \qquad \nabla x = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \qquad \nabla y = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$
Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

90. Let $P = \sqrt{u^2 + v^2 + w^2}$, where $u = xe^{-y}$, $v = ye^x$, and $w = \frac{xy}{e^x}$. Find $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ using the chain rule.

91. A right circular cone is being expanded by animation software. It currently has sliders set for radius 25 and height 60 cm. Let r be the radius, h the height, and V the volume.
   (a) If height is increasing at 5 cm/sec, but the radius is holding constant, find $\frac{dV}{dt}$.
   (b) If height is increasing at 5 cm/sec, but the radius is decreasing by 2 cm/sec, find $\frac{dV}{dt}$.
   (c) If height is increasing at 5 cm/sec, and volume is holding constant, find $\frac{dr}{dt}$.
   (d) If height is increasing at 5 cm/sec, and volume is increasing by 40 cm^3/sec , find $\frac{dr}{dt}$.

92. There is a circular (as viewed from above) path centered at the origin with radius 800 ft. Let $z = f(x, y)$ be the ground's elevation at a particular point, and suppose
$$\nabla f(x, y) = 10^{-11} \begin{bmatrix} 3x^2y + 10xy^2 \\ x^3 + 10yx^2 \end{bmatrix}$$
You observe someone start walking from the southernmost point on the path, going clockwise at 5 ft/sec (apparently from above). The walker's elevation undulates with the surface of the ground.
   (a) Write parametric equations for the x and y coordinates of location, for $t \in [0, \infty)$.
   (b) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
   (c) Use the multi-variable chain rule to find $\frac{dz}{dt}$ when $t = 180$ seconds.
   (use an Octave script to manage the variables.)

Integration

93. Find the EXACT value of a such that $\displaystyle\int_2^5 \left(\int_0^a \frac{x}{y} dx \right) dy = 3$.

94. Let $R = [3, 8] \times [1, 4]$. Use WolframAlpha to evaluate $\displaystyle\iint\limits_R e^{-xy/16} dA$.

95. Let $R = [3, 8] \times [1, 4]$. Consider the integral $\displaystyle\iint\limits_R x \sin(xy) dA$. Write the integral with $dA = dxdy$, and also write it with $dA = dydx$. Evaluate by hand whichever is easier.

96. Suppose a pond has a surface area of 3200 square feet, and an average depth of 3.64 feet. Let $d(x, y)$ be the depth at a given point in the pond's region R. Find $\iint_R d(x, y)dA$.

97. Let R be bounded by $|x - 12| = 8$ and $|y - 9| = 7$. Find the average distance of points in R to the center of that rectangle. (Do the integral in WA)

98. Let $f(x, y) = \dfrac{5x^3}{(4 + x^2y)^{3/2}}$. Do these integrals by hand (with dx on the outside).

    (a) Do an improper integral over the horizonal strip $[0, \infty) \times [0, 1]$. Does it converge?

    (b) Do an improper integral over the vertical strip $[0, 1] \times [0, \infty)$. Does it converge?

    (c) Integrate over the square $[0, a] \times [0, a]$. Your answer should be a function of a.

    (d) Find the value of a that makes that last integral evaluate to 1.

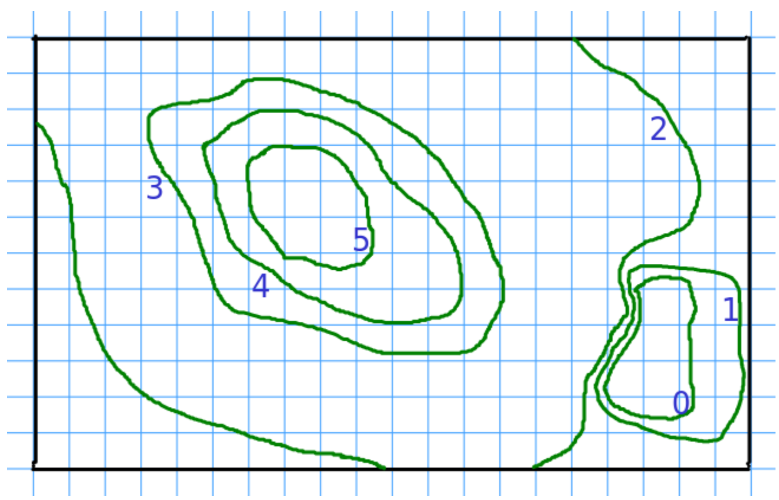
99. Let $f(x, y) = xe^{(y-x^2)/5}$, and let $R = [0, a] \times [0, b]$.

    (a) Show that this function is **separable**, i.e. $f(x, y) = g(x)h(y)$.

    (b) Write $\iint_R f(x, y)dA$ as a function $V(a, b)$.

    (c) Find $\nabla V(a, b)$ where $a = 1$ and $b = 3$.

100. The population density of a 20×12 km city is shown in this contour plot. Densities are in thousands of people per square km. Use a Riemann sum with 4×2 blocks to estimate the city's population. (Notice the lake in the SE corner where density drops to zero.



101. Evaluate $\int_0^2 \int_{y^2}^{2y} (4x - y)dxdy$ by hand

102. Sketch the region of integration, reverse the order of integration, and evaluate by hand

$$\int_0^1 \int_{2x}^2 e^{y^2} dydx$$

103. Evaluate $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy$ by hand.

104. Integrate $f(x, y) = e^x \ln y$ over the region in the 1st quadrant, above $x = \ln y$, having $|y - 3| < 2$. Do the inner integral by hand, but use W.A. for the outer integral.

105. Consider a circle of diameter D, described by $r = D\sin(\theta)$. Let P be a point on the circle's boundary. Find the average distance from a point inside the circle to the point P.

106. A swimming pool has a boundary described by the cardioid

$$r = 15(1 + \cos\theta)$$

The bottom of the pool slopes linearly from a depth of 3 feet when $x = 0$, to a depth of 12 feet when $x = 30$. Set up the integral calculations to find the volume and average depth of the pool.

107. Sketch the region bounded by $y = 4x - x^2$ and $y = -x$. Set up the integrals necessary to find

 (a) Area
 (b) Centroid (center of mass if density is constant)
 (c) Mass if density $\delta(x, y) = xy$
 (d) Center of mass with that density.

108. A rectangular city covers the region $[0, 10] \times [0, 6]$; say units are miles. There are two hospitals: one at $(3, 1)$ and the other at $(9, 5)$.

 (a) Sketch the region, hospitals, and dividing line for determining which is closer.
 (b) Find the average distance (as the crow flies) from all points in the city to the closer hospital.

109. Now suppose the 10×6 city has population density $\delta(x, y)$ that decreases linearly as you move away from the city's center. The density is 1000 people per square mile at the center, and 100 at the corners.

 (a) Write a formula for $\delta(x, y)$.
 (b) Find the population density at $(3, 4)$.
 (c) Find the total population of the city.
 (d) Find the average population density for the city.
 (e) Suppose the hospital at $(9, 5)$ closes, leaving only one hospital at $(3, 1)$. Accounting for population density, what is the average distance (as the crow flies) to the hospital.
 (f) There is a Pal's located at in the north at $(7, y)$. It is an average of 4 miles from the population. Use trial-and-error to find y to the nearest 0.01.

110. The region R is bounded by $y = \ln(x)$, $y = -x$, and $y = 2$. Use guess-check, or your calculator to find the intersections. Set up double integral calculations for the following:

  (a) area

  (b) centroid

  (c) average distance to the origin

111. The region R is bounded below by $y = \frac{1}{2}|x|$ and above by $x^2 + y^2 = 100$. Let $f(x, y) = x^2y + y^3$ represent either height or density. Set up double integral calculations in polar coordinates for the following:

  (a) area A

  (b) volume V (interpreting f as height)

  (c) average height

  (d) surface area of f above R

  (e) mass M (now interpreting f as density)

  (f) average density

  (g) center of mass $(\overline{x}, \overline{y})$

  (h) average distance (uniform density) to the origin

  (i) average distance (weighted by density f) to the origin

112. The region R is bounded by $y = x(b - x)$ and the y-axis. If $\overline{y} = 5$, find $\overline{x}$.

113. A disc is in the shape of a unit circle, and is made out of a material with density δ inversely proportional to the distance from the center. If the mass is $M = 20$, then solve $\delta(0.3, y) = 5$.

114. Set up an integral to find the surface area of $f(x, y) = x^2 \sin(y/2)$ lying above the triangle with corners at $(0, 0)$, $(2, 5)$, and $(7, 0)$.

115. An infinite city occupies the region below $y = e^{-kx^2}$ in the first quadrant. If the population density is $\frac{x}{\sqrt{y}}$, find the total population as a function of k. Find the value of k such that the population is 25.