Warm-Up

1. $f(x) = x^2$ has average height of 100 over the interval $[5, b]$. Find the "exact" value of b "by hand". Follow the Ten Commandments, and include an illustrating sketch.

Polar Coordinates

2. (Apex 9.4) 5,7,25,31,37,51
3. Convert $\left(\frac{5\pi}{12}, 20\right)_P$ to Cartesian coordinates.
4. Convert $(-4, 7)_C$ to polar coordinates.
5. Convert $(\frac{\pi}{2}, -6)_C$ to polar coordinates, and use degrees to measure the angle.
6. List 4 aliases of $(\frac{\pi}{4}, 2)_P$.
7. How can you describe all aliases of $(\frac{\pi}{3}, 7)_P$? (Hint: use expressions involving an integer k)
8. Sketch the region described by
   (a) $r \in (2, 3]$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.
   (b) $|r| \leq 3$ and $|\theta| \leq \frac{\pi}{4}$
9. Find a Cartesian equation and describe the graph.
   (a) $\theta = \frac{\pi}{3}$
   (b) $r = 2\cos\theta$
   (c) $r = \sec(\theta)\tan(\theta)$
10. Find a polar equation and describe the graph.
   (a) $x^2 + y^2 = 25$
   (b) $y = 1$
   (c) $x + 2y = 10$
11. Show that $r = a\sin\theta + b\cos\theta$ is a circle. Find the center and radius.
12. The curve $r = \sin(\theta)\tan(\theta)$ is called a "cissoid". Prove that the graph lies inside the vertical strip $x \in [0, 1)$.
13. Use Desmos to investigate the family of curves

$$r = \frac{1 - a\cos(\theta)}{1 + a\cos(\theta)}$$

   Find transitional values of a where the curve changes its basic shape.
14. Sketch the polar curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$ on the same graph, and determine where they intersect in Cartesian coordinates.
15. Sketch $r = 1$ and $r = 2\cos\theta$ on the same graph. Label the intersections. Shade the region bounded between the curves.

Parametric Equations

16. (Apex 9.2) 8,11,13,17,19,25,37,39

17. Write using equations parameterized by t, and sketch the graph.

    (a) $y = e^{-x}$, $x \in [-2, 2]$

    (b) $r = e^{-\theta}$, $\theta \geq 0$

18. Parameterize a circle with area 154 that is centered at $(0, 12)$. For $t \in [0, \infty)$ minutes, the position should start at the top of the circle and move counter-clockwise with a frequency of 20 RPM.

19. Parameterize this ellipse for $t \geq 0$ seconds.

$$\left(\frac{x-6}{4}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

    starting at $(2, 2)$, moving clockwise, with a period of 24 seconds.

20. Plot points to sketch parametric equations $(e^{2t}, t+1)$ for $t \in \mathbb{R}$. Then eliminate the parameter to write y as a function of x, and state the domain.

21. Do this matching problem from the Stewart textbook:

28. Match the parametric equations with the graphs labeled I-VI. Give reasons for your choices. (Do not use a graphing device.)

(a) $x = t^4 - t + 1$, $y = t^2$

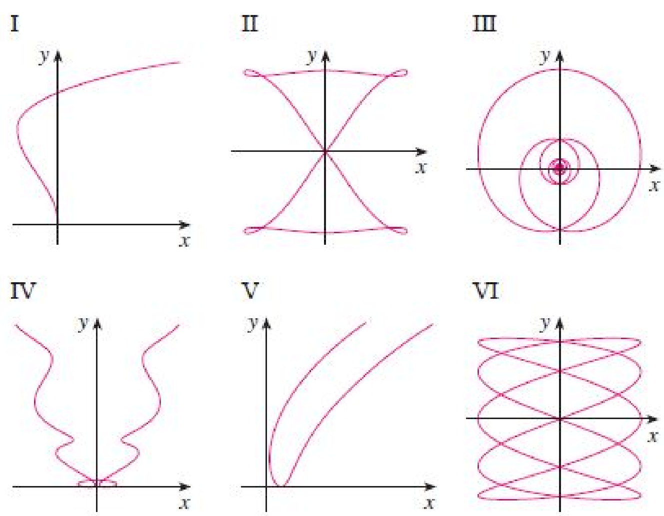
(b) $x = t^2 - 2t$, $y = \sqrt{t}$

(c) $x = \sin 2t$, $y = \sin(t + \sin 2t)$

(d) $x = \cos 5t$, $y = \sin 2t$

(e) $x = t + \sin 4t$, $y = t^2 + \cos 3t$

(f) $x = \dfrac{\sin 2t}{4 + t^2}$, $y = \dfrac{\cos 2t}{4 + t^2}$



22. Use Desmos to investigate the "Lissajous figures" described by $x = a\sin(nt)$ and $y = b\cos(t)$ for $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$.

23. Suppose $(x(t), y(t))$ are parametric equations that describe motion that covers a path for $t \in [0, 12]$. Modify the equations to:

   (a) Shift the path 4 units north, and 7 units west.

   (b) Shift the path 9 units at an angle of θ radians counter-clockwise from east.

   (c) Move twice as fast to cover the path for $t \in [0, 6]$.

   (d) Move backwards to cover the path for $t \in [0, 4]$.

   (e) Double the size of the path, while maintaining its shape and starting point.

   (f) Bonus: rotate the path θ radians counter-clockwise around the origin.

24. Let $y = \frac{1}{1+x}$ for $x \geq 0$. The obvious parameterization is $\left(t, \frac{1}{1+t}\right)$ for $t \geq 0$. Re-parameterize in such a way that the same path is traced for $t \in [0, 1)$.
Hint: let x be a one-to-one function with domain $[0, 1)$ and range $[0, \infty)$

25. Parameterize the line that goes from $(3, 15)$ to $(11, 9)$ in 5 seconds.

Parametric/Polar Calculus

26. (Apex 9.3) 7,15,38 (use Desmos)
27. You have parametric equations with $x(3) = 75, y(3) = 42, \frac{dx}{dt}(3) = -2, \frac{dy}{dt}(3) = 6$.
    (a) Find the equation of the tangent line when $t = 3$.
    (b) Estimate $x(3.5), y(3.5)$.
    (c) Estimate $x(2.7), y(2.7)$.
28. Consider these parametric equations that trace a curve C.

$$x = t(5-t), \qquad y = t(5-t)(3-t), \qquad t \in [0, 5]$$

Let R be the region enclosed by C.

    (a) Sketch the graph, and use arrows to indicate the orientation of C.
    (b) Find the equation of the tangent line when $t = 1$.
    (c) Find the eastern-most point of R.
    (d) Find the northern-most point of R.
    (e) Find the southern-most point of R.
    (f) Write the integral to find the perimeter of R, and use software to evaluate.
    (g) Write the integral to evaluate the area of R, and evaluate it by hand.
    (h) What percentage of the area is above the x-axis?
    (i) Find the vertical line that splits the area of R 50-50.
29. Find the tangent line to the graph described parametrically by $\left(\frac{20}{1+t^2}, 5(.8)^t\right)$ when $t = 2$.
30. (Apex 9.5) 9,13,19,27,31
31. Find the length of these curves (set up integral; evaluate using software).
    (a) $y = xe^{-x}$ for $x \in [0, 2]$
    (b) $r = \sin(2\theta)$ for $\theta \in [0, \frac{\pi}{2}]$
    (c) $(\sin(t), e^{-t^2})$ for $t \in [-\pi, \pi]$
32. Find the area of the bounded region (set up integral; evaluate using software).
    (a) $r = \sin(2\theta)$ for $\theta \in [0, \frac{\pi}{2}]$
    (b) $(\sin(t), e^{-t^2})$ for $t \in [-\pi, \pi]$
33. Find the perimeter and area of the region bounded by x-axis and the parametric curve:

$$x = (t-1)^2$$

$$y = (t-2)(t-5)$$

34. Consider $r = \exp(\theta)$ for $\theta \in [0, 2\pi]$.

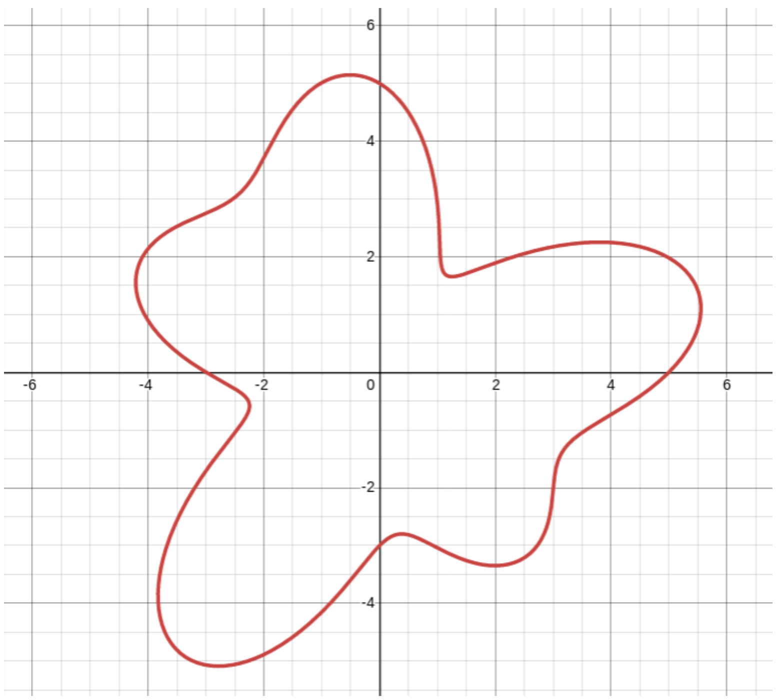
(a) Sketch the graph.
(b) Find points where the graph is horizontal.
(c) Find points where the graph is vertical.

35. Let $r = \sin(2\theta)$.

(a) Describe the shape of the graph.
(b) Find the expression for $\frac{dy}{dx}$ in terms of θ (don't simplify).
(c) Find the slope of the graph when $\theta = \frac{\pi}{6}$.
(d) Find the equation of the tangent line at that point.

36. Suppose $r(\frac{\pi}{8}) = 17$ and $\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{8}} = 2.68$. Use differentials to estimate $x(\frac{\pi}{9})$ and $y(\frac{\pi}{9})$.

37. The zoo train track follows the polar curve $r = 4 + \cos(3\theta) + \sin(5\theta)$, pictured here.



(a) Find the length of the track (set up integral, evaluate with software).
(b) Find the enclosed area (set up integral, evaluate with software).
(c) Find the area north of the x-axis.
(d) Find the area west of the y-axis.
(e) Find the point on the track closest to the origin (hint: use calculus to minimize r).

Vectors, Space, Dot, Cross

38. (Apex 10.2) 11,17,27,28,35 (see ex. 10.2.8)

39. Find a vector with length 20, that points in the opposite direction of $\vec{v} = [5, -3, 9]$.

40. Write the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ associated with the polar point $(5, \frac{\pi}{8})_P$ in $\mathbb{R}^2$.

41. Consider points $P(1, 8, -2)$ to $Q(3, 2, 7)$ in $\mathbb{R}^3$.
    (a) find $\text{dist}(P, Q)$
    (b) write a vector that points from P to Q
    (c) write a vector that points from Q to P
    (d) write a vector that points from P through Q, and then another 1 unit beyond
    (e) write a vector that points from the origin to the midpoint of $\vec{PQ}$
    (f) find the equation of the sphere centered at P with volume 100 cubic units
    (g) find the equation of the sphere with $\vec{PQ}$ as a diameter

42. Find two unit vectors in $\mathbb{R}^2$ parallel to the tangent line of $y = e^{-x^2}$ at $x = 1$.

43. True or false? Justify your answer.

$$\|\vec{a} - \vec{b}\| = \|\vec{a}\| - \|\vec{b}\|$$

44. This problem is from a physics book.

85 You are kidnapped by political-science majors (who are upset because you told them political science is not a real science). Although blindfolded, you can tell the speed of their car (by the whine of the engine), the time of travel (by mentally counting off seconds), and the direction of travel (by turns along the rectangular street system). From these clues, you know that you are taken along the following course: 50 km/h for 2.0 min, turn 90° to the right, 20 km/h for 4.0 min, turn 90° to the right, 20 km/h for 60 s, turn 90° to the left, 50 km/h for 60 s, turn 90° to the right, 20 km/h for 2.0 min, turn 90° to the left, 50 km/h for 30 s. At that point, (a) how far are you from your starting point, and (b) in what direction relative to your initial direction of travel are you?

45. Consider these spheres:
    - $(x - 3)^2 + y^2 + z^2 = 25$
    - $x^2 + y^2 + z^2 = 24x - 10y + 16z - 229$

    (a) Find the center and radius of each sphere.
    (b) Find the distance between the spheres.
    (c) Find an equation describing the set of all points equidistant from the center of each sphere.
    (d) Plot that equation from part (c), along with both spheres, in Geogebra3D.

46. Three points are **co-linear** if they lie in the same line. Let $P(4, 2, 7)$, Q, and R be co-linear. Find coordinates for Q and R if $\vec{PQ} = [-2, 5, 3]$ and $\vec{QR} = -1.5\vec{PR}$.

47. Let $(x-6)^2 + (y-1)^2 + (z-5)^2 = 75$ be a sphere.

    (a) Is the point $P(9, -4, 11)$ inside, outside, or on the sphere ?

    (b) Find two points where the sphere intersects the x-axis.

    (c) Find the distance from $Q(2, 10, 8)$ to the sphere.

48. There is a sphere S_1 with equation $(x-5)^2 + (y-7)^2 + (z-4)^2 = 9$. Find the equation of another sphere S_2 with:

    - 75% more volume than S_1
    - center co-linear with the origin and S_1's center
    - 10 units distance between S_1 and S_2

49. Let $P(4,7,3t)$ and $Q(t,0,t^2)$ be two points parameterized by t.
   (a) Write $\text{dist}(P,Q)$ as a function of t. Graph it in Desmos.
   (b) Evaluate that function at time $t=5$.
   (c) At what positive time is $\text{dist}(P,Q)=10$?
   (d) Set $\frac{d}{dt}\text{dist}(P,Q)=0$ to find the time at which P is closest to Q. How close do they get ?
50. (Apex 10.3) 15,23,29,33
51. Find the angle (in degrees) between vectors $[1,6,4]$ and $[2,5,1]$.
52. Find c so that $[1,c,3]$ and $[-2,2,4]$ are orthogonal.
53. Let $\theta(a)$ be the angle (in degrees) between the vectors $[2,a,7]$ and $[a^2,1,5]$.
   (a) Find $\theta(3)$.
   (b) Find $\left.\frac{d\theta}{da}\right|_{a=3}$.
54. If $\|\vec{u}\|=7$, $\|\vec{v}\|=12$, and the sine of the angle betweeen them is 0.75, then find two possible values for $\vec{u}\cdot\vec{v}$. What is the smallest possible value of $\|\vec{u}-\vec{v}\|$?
55. Find values of a such that the angle between $[2,1,-1]$ and $[1,a,0]$ is $45°$.
56. Find the acute angle between $y=x^2$ and $y=x^3$ where they intersect at $(1,1)$.
57. $\|\vec{PQ}\|=19$, $\|\vec{RQ}\|=26$, and $\vec{PQ}\cdot\vec{RQ}=494$. Explain how you know P,Q,R are co-linear. Find $\text{dist}(P,R)$.
58. Find the acute angle (in degrees) between $\vec{v}=[1;1;1]$ and
   (a) the y-axis
   (b) the xy-plane
59. Find the unit vectors in $\mathbb{R}^2$ that make a $20°$ angle with the vector $[1,7]$.
60. Show that if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.
61. (Apex 10.4) 19,23,31,35
62. Consider points $P(7,0)$, $Q(0,5)$ and $R(a,a)$.
   (a) Sketch the triangles for $a=2$ and $a=9$ on the same graph.
   (b) Find the perimeter of $\triangle PQR$ as a function $P(a)$.
   (c) Find the area of $\triangle PQR$ as a function $A(a)$.
   (d) Find $\frac{dA}{da}$
   (e) For what value of a is the area zero ?
   (f) Embed $P(7,0,0)$ and $Q(0,5,0)$ into $\mathbb{R}^3$, and raise the third point to $R(a,a,a)$. Now find the value of a that minimizes the area of $\triangle PQR$.
63. If $\vec{v}\cdot\vec{w}=6$ and $\|\vec{v}\times\vec{w}\|=11$, find the angle (in degrees) between $\vec{v}$ and $\vec{w}$.
64. Consider points $P(1,3,5)$, $Q(2,4,8)$, $R(4,9,11)$, and $S(x,7,2)$.
   (a) Let $x=6$ and find the volume of the parallelpiped with adjacent corners at P,Q,R,S.
   (b) Find the value of x so that S is co-planar with ΔPQR.

65. A parallelpiped determined by adjacent corners: $P(0,0,0)$, $Q(3,1,2)$, $R(-1,4,1)$, $S(2,5,z)$, has volume 100.

(a) If $z > 0$, find its value.

(b) If $z < 0$, find its value.

66. A molecule of methane, CH_4, is structured with four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by $H - C - H$ combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about $109.5°$.

    Hint: Take the vertices of the tetrahedron to be the points $(1,0,0), (0,1,0), (0,0,1)$, and $(1,1,1)$. Then the centroid is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

67. The Gordon line ($y = x$) intersects the curve $y = \sqrt{x}$ at $(0,0)$ and $(1,1)$. Find the acute angle (in degrees) of intersection at each of those points.

Lines, Planes

68. (Apex 10.5) 9,11,17

69. (Apex 10.6) 9,11,13,19,21,25

70. Find the equation of the plane through $(5,8,3)$ and orthogonal to the vector $[7,-2,4]$.

71. A sphere is centered at $(7,-2,15)$ and the vector $\vec{v} = [9,13,5]$ goes from the center to the surface of the sphere.
    (a) Find the equation of the sphere.
    (b) Find the equation of the axial line through the sphere's center in the direction of $\vec{v}$.
    (c) Find the equation of the plane, normal to $\vec{v}$, that splits the sphere into two hemispheres.

72. Let $P(2,4,9)$, $Q(7,3,5)$, and $R(1,8,2)$ be three points in $\mathbb{R}^3$.
    (a) Find the equation of the plane containing those points.
    (b) Find the equation of the line through P that is perpendicular to the plane.
    (c) Find the area of $\triangle PQR$.

73. Consider the line $\vec{\ell}(t) = \begin{bmatrix} 3+5t \\ 1-7t \\ 8+2t \end{bmatrix}$ and the plane $z - 15 = 2(x-3) + 9(y+1)$.
    (a) Find the point of intersection.
    (b) Find the acute angle (in degrees) of intersection.

74. Consider the line $\vec{\ell}(t) = \begin{bmatrix} 3+5t \\ 1-7t \\ 8+2t \end{bmatrix}$ and the sphere $(x-3)^2 + (y+1)^2 + (z-15)^2 = r^2$.
    (a) Find the values of t, and the correspondings point of intersection if $r = 10$.
    (b) Find the EXACT value of r such that line is tangent to the sphere (one intersection). Then find that EXACT point of tangency.
    (c) Using a computer, find the value of r such that the points of intersection are 50 units apart.

75. These two lines are known to intersect. What is the value of c ? Follow these steps:

$$\vec{\ell}_1(t) = \begin{bmatrix} 4t \\ 2t-7 \\ 3t-2 \end{bmatrix} \qquad \vec{\ell}_2(t) = \begin{bmatrix} 14-6t \\ 12+t \\ 5+ct \end{bmatrix}$$

(a) Using different symbols for the parameters, e.g. t_1 and t_2, set the x and y coordinates equal and find the point of intersection. (you are ignoring the z-coordinates for now)

(b) Now set the z coordinates equal to find the value of c.

76. Find the line of intersection between these planes. Follow the steps:

$$z + 80 = 2(x-3) + 5(y+1) \qquad\qquad 4x + y + 3z = 45$$

(a) Set $x = 0$ and solve for y and z to find one point on the line.

(b) Set $z = 0$ and solve for x and y to find one point on the line.

(c) Write parametric equations of the line through those two points.

77. Find the value of z so that the point $(8, 2, z)$ is on the plane $3(x-8) + 5(y-2) + 9(z-1) = 36$.

78. Find an equation for the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$.

79. Find a vector $\vec{v}$ that satisfies these conditions:

- $\|\vec{v}\| = 12$
- $\vec{v} \cdot \vec{k} = 1$
- parallel to the plane $3x - 5y + 7z = 74$

80. Consider the plane with x, y, and z intercepts of 24, 6, and 15 respectively.

(a) Find the equation of the plane.

(b) Find the equation of the line through the origin and the "centroid" of the triangle of intercepts.

(c) Find the angle (in degrees) at which that line pierces the plane.

Multi-variable Functions

81. (Apex 12.1) 11,13,19,21
82. (Apex 12.3) 5,9,11,15,17,27,29,31
83. Let $z = x^2 - y$. The contour that contains (3,2) also passes through $(8, y)$ for what value of y?
84. Let $z = f(x, y) = \exp(-x^2 - y^2)$.
    (a) Find the range.
    (b) Find the radius of the circular contours for $z = .5$, $z = .2$, and $z = .1$.
85. Let $z = f(x, y) = \log_2(xy + 1)$.
    (a) Find the domain and shade it on the xy-plane.
    (b) Sketch the contours corresponding to $z = -2$, $z = -1$, $z = 0$, $z = 1$, and $z = 2$.
86. Contrast the contours of $z = 2x + y$ and $z = \sin(2x + y)$.
87. Let $f(x, y) = |x| + |y|$. For this problem, consider contours for $z = 0, 1, 2, 3, 4, 5$
    (a) Sketch the contours of $z = f(x, y)$. Describe the shape and spacing.
    (b) Sketch the contours of $z = f^2(x, y) = (|x| + |y|)^2$. Describe the shape and spacing.
    (c) Sketch the contours of $z = \sqrt{f(x, y)} = \sqrt{|x| + |y|}$. Describe the shape and spacing.
88. (Apex 12.4) 5,9,19,21
89. (Apex 12.6) 7,13,19
90. (Apex 12.7) 9,13,17,23
91. Here is a table of points for a function $z = f(x, y)$.

x	y	z
7.00	3.85	14.50
7.03	4.00	14.90
6.97	4.00	15.40
7.00	4.15	16.10

    (a) Average the surrounding points to estimate $f(7, 4)$.
    (b) Use difference quotients to estimate $\nabla f(7, 4)$.
    (c) Use those answers to write an approximate tangent plane at $(7, 4)$.
    (d) Estimate $f(7.05, 4.09)$.
92. If $dz = 2dx + 5dy + 4dt$, find the equation of the tangent plane at $(7, 0, 3, 14)$.
93. You have a company that sells snowshovels. Quantity sold (S) is a function of price (P dollars), temperature (T degrees), and advertising (A units). Suppose $dS = -25dP - 7dT + 4dA$.
    (a) Find $\frac{\partial S}{\partial P}$, and write a sentence that explains its sign.
    (b) Find $\frac{\partial S}{\partial T}$, and write a sentence that explains its sign.
    (c) Find $\frac{\partial S}{\partial A}$, and write a sentence that explains its sign.
    (d) If temperature falls 10 degrees, you raise the price by a dollar, and reduce advertising by 6 units, estimate the net affect on S.

94. A bird is flying south for the winter. Let $T(t, x, y)$ be the expected daily high temperature as a function of time in days, degrees east longitude, and degrees north latitude. At the current time and location, suppose $T = 62$ and $\nabla T = [-0.18, .07, -1.53]$. Over the next week, the bird hopes to move $2°$ east and $5°$ south. Find the differential dT.

95. Let $f(x, y) = \frac{5x}{1+y^2}$.

    (a) Evaluate $f(9, 3)$.
    (b) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as functions of x and y.
    (c) Evaluate the gradient $\nabla f(9, 3)$.
    (d) Find the equation of the tangent plane at $(9, 3)$.
    (e) Find the equation of the normal line.
    (f) Using the linearization (i.e. tangent plane) estimate $f(9.17, 2.88)$.
    (g) Evaluate $f(9.17, 2.88)$ exactly using the non-linear formula for f.
    (h) Find the directional derivatives $D_{\vec{v}}f(9, 3)$ for these directions:

        i. $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
        ii. $\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
        iii. $\vec{v} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$
        iv. $\vec{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$
        v. $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$
        vi. $\vec{v} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$
        vii. $\vec{v} = \begin{bmatrix} -5 \\ -12 \end{bmatrix}$
        viii. $\vec{v} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

96. Consider a multivariable function with $f(29, 52) = 73$ and $\nabla f(29, 52) = \begin{bmatrix} -0.23 \\ 0.09 \end{bmatrix}$.

    (a) Find the equation of the tangent plane at the given point.
    (b) Estimate $f(29.5, 50.2)$
    (c) Find the equation of the normal line.
    (d) Let $\vec{v} = [a; 7]$. If $a = -2$, find $D_{\vec{v}}f(29, 52)$.
    (e) Find the value of a such that $D_{\vec{v}}f(29, 52) = 0$

97. Suppose at a certain point, $\nabla f = \begin{bmatrix} 0.21 \\ -0.56 \end{bmatrix}$ and the Hessian is $H = \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$. Consider the velocity direction $\vec{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

    (a) Find the slope: $D_{\vec{v}}f$.
    (b) Find the concavity: $D_{\vec{v}}(D_{\vec{v}}f)$.
    (c) If you move in the opposite direction, $-\vec{v}$, then what are the slope and concavity ?
    (d) Find the steepest ascent direction, and the slope in that direction.
    (e) Find the steepest descent direction, and the slope in that direction.
    (f) Find a direction where the slope is zero.
    (g) Find a direction where the slope is 0.35.
    (h) Find a direction where the slope is -0.35.

98. Let $f(x, y) = x^3 + \frac{15x}{y}$

    (a) Find the gradient $\nabla f(x, y)$; then evaluate $\nabla f(2, 5)$.
    (b) Find the Hessian matrix $H(x, y)$; then evaluate $H(2, 5)$.

(c) Find the equation of the tangent plane at $(2,5)$.

(d) Find the equation of the normal line at $(2,5)$.

99. This plane lies tangent to the surface of $f(x,y)$.

$$z = 425 + 3(x-90) + 7(y+260)$$

(a) Find the evident point P of tangency.

(b) Find ∇f at P.

(c) Find the slope of the surface in the steepest ascent direction.

(d) Estimate $f(86,-257)$.

(e) Find the equation of the line through P orthogonal to the surface of f.

(f) Find the equation of the tangent line to the contour $f(x,y) = 425$.

100. Suppose at a certain point, the surface of a function has

- slope 7 in the direction $[3,4]$
- slope 1 in the direction $[8,15]$

Find ∇f.

101. Suppose $f(200,700) = 125$, $f(201,704) = 143$, and $f(198,703) = 119$.

(a) Estimate $\nabla f(200,700)$.

Hint: $f(200+dx, 700+dy) - f(200,700) \approx \nabla f(200,700) \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$

(b) Estimate $f(203,695)$.

102. Suppose you are on a hill and the steepest descent direction is east by southeast (i.e. $\theta = -11.25°$), having slope -0.35. Find the slope in the southwest direction ($\theta = 225°$).

103. Consider the surface of $f(x,y) = xy$ and the line $\ell(t) = \begin{bmatrix} 1+2t \\ 2+3t \\ 146-23t \end{bmatrix}$.

(a) Find the points of intersection.

(b) Find the acute angles (in degrees) at which the line intersects the surface at those points.

104. Let $f(x,y) = \frac{x^2}{y}$.

(a) Find the equation of the tangent plane at $(6,4)$.

(b) Find the slope of the surface in the direction of $\vec{v} = [8,-5]$.

i. At an arbitrary point (x,y).

ii. At $(6,4)$.

(c) Find the concavity in the direction of $\vec{v}$.

i. At an arbitrary point (x,y).

ii. At $(6,4)$.

105. Find all points at which the direction of fastest change of $f(x,y) = x^2 + y^2 - 2x - 4y$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

106. Show that $v = (x - at)^4 + \cos(x + at)$ satisfies the wave equation:

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$$

107. Show that $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ satisfies the Laplace equation:

$$f_{xx} + f_{yy} = 0$$

108. Let $f(x, y) = \ln(10 + x^2 + 3x + y^2)$
   (a) Find $\nabla f(x, y)$, then evaluate $\nabla f(2, 5)$.
   (b) Find the Hessian matrix $H(x, y)$.
   (c) Let $(2, 5)$ be your current location, and consider walking in the walking in the 11:20 clock direction.
      i. Write a unit vector representing your direction (use cos/sin of your angle).
      ii. Find the slope.
      iii. Find the concavity using the formula $\vec{v} \cdot (H\vec{v})$.

Chain Rule

109. (Apex 12.5) 11,17,21,29

110. Let z be a function of x and y, and let both x and y be functions of s and t. At a certain point,

$$\nabla z = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \qquad \nabla x = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \qquad \nabla y = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

   Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

111. Let $P = \sqrt{u^2 + v^2 + w^2}$, where $u = xe^{-y}$, $v = ye^x$, and $w = \frac{xy}{e^x}$. Find $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ using the chain rule.

112. A right circular cone is being expanded by animation software. It currently has sliders set for radius 25 and height 60 cm. Let r be the radius, h the height, and V the volume.
   (a) If height is increasing at 5 cm/sec, but the radius is holding constant, find $\frac{dV}{dt}$.
   (b) If height is increasing at 5 cm/sec, but the radius is decreasing by 2 cm/sec, find $\frac{dV}{dt}$.
   (c) If height is increasing at 5 cm/sec, and volume is holding constant, find $\frac{dr}{dt}$.
   (d) If height is increasing at 5 cm/sec, and volume is increasing by 40 cm^3/sec , find $\frac{dr}{dt}$.

113. There is a circular (as viewed from above) path centered at the origin with radius 800 ft. Let $z = f(x, y)$ be the ground's elevation at a particular point, and suppose

$$\nabla f(x, y) = 10^{-11} \begin{bmatrix} 3x^2y + 10xy^2 \\ x^3 + 10yx^2 \end{bmatrix}$$

   You observe someone start walking from the southernmost point on the path, going clockwise at 5 ft/sec (apparently from above). The walker's elevation undulates with the surface of the ground.
   (a) Write parametric equations for the x and y coordinates of location, for $t \in [0, \infty)$.
   (b) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
   (c) Use the multi-variable chain rule to find $\frac{dz}{dt}$ when $t = 180$ seconds.
   (use an Octave script to manage the variables.)

114. (Apex 12.5) 23,25
(Apex 12.7) 21,23

115. Consider the graph of $\exp(y/z) = xz^2$ as defining z as an implicit function of x and y. At the point where $x = 0.5$ and $y = 1.2$, find numerical values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

116. Suppose z is an implicit function of x and y, with $y(z^2 + 2) = xe^z$.

    (a) Verify that $(6, 3, 0)$ is on the graph.

    (b) Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by differentiating implicitly.

    (c) Find $\nabla z(6, 3, 0)$.

    (d) Find the equation of the tangent plane at that point.

    (e) Find the equation of the normal line at that point.

117. Find the quadratic function $f(x, y) = c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 y^2 + c_6 xy$ such that ("exactly"):

    (a) $f(5, 2) = 10$

    (b) $\nabla f(5, 2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

    (c) $\frac{\partial^2 f}{\partial x^2} = 0.70$

    (d) The concavity is 0.20 in the direction $\vec{v} = \begin{bmatrix} -10 \\ 3 \end{bmatrix}$.

    (e) The concavity is 0.20 in the direction $\vec{v} = \begin{bmatrix} -10 \\ 6 \end{bmatrix}$.

    Hint: start by matching the 2nd derivatives.

118. (Apex 12.8) 5,9,13

119. This function has a local max and a local min. How much higher is the local max than the local min?

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 + 20$$

120. Suppose the gradient of a function is:

$$\nabla f = \begin{bmatrix} y(y-6)(x+1) \\ x^2 y - 3x^2 + 2xy - 6x - 5.5y + 9 \end{bmatrix}$$

Find and classify the critical points (there are 5 of them)

121. Consider the plane with x, y, and z intercepts of 2, 5, and 3 respectively. There is a point $P(3, 4, 6)$ that is not on the plane.

    (a) Find the equation of the plane.

    (b) Write the objective function $f(x, y)$ giving the squared distance from P to a point (x, y, z) on the plane.

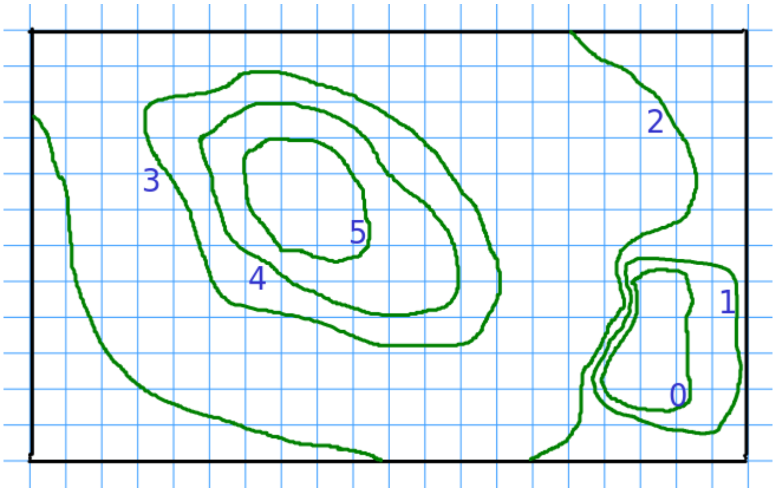
    (c) Minimize f by setting $\nabla f = \vec{0}$, and verify that it's a minimum with the 2nd derivative test.

    (d) Find the shortest distance from P to the plane.

122. Find the line $y = a + bx$ that "best" fits the points $\{(0, 2), (5, 0), (3, 3)\}$. The objective function should be the sum of squared y errors as demonstrated in class.

Integration

123. The population density of a 20×12 km city is shown in this contour plot. Densities are in thousands of people per square km. Use a Riemann sum with 4×2 blocks to estimate the city's population. (Notice the lake in the SE corner where density drops to zero.

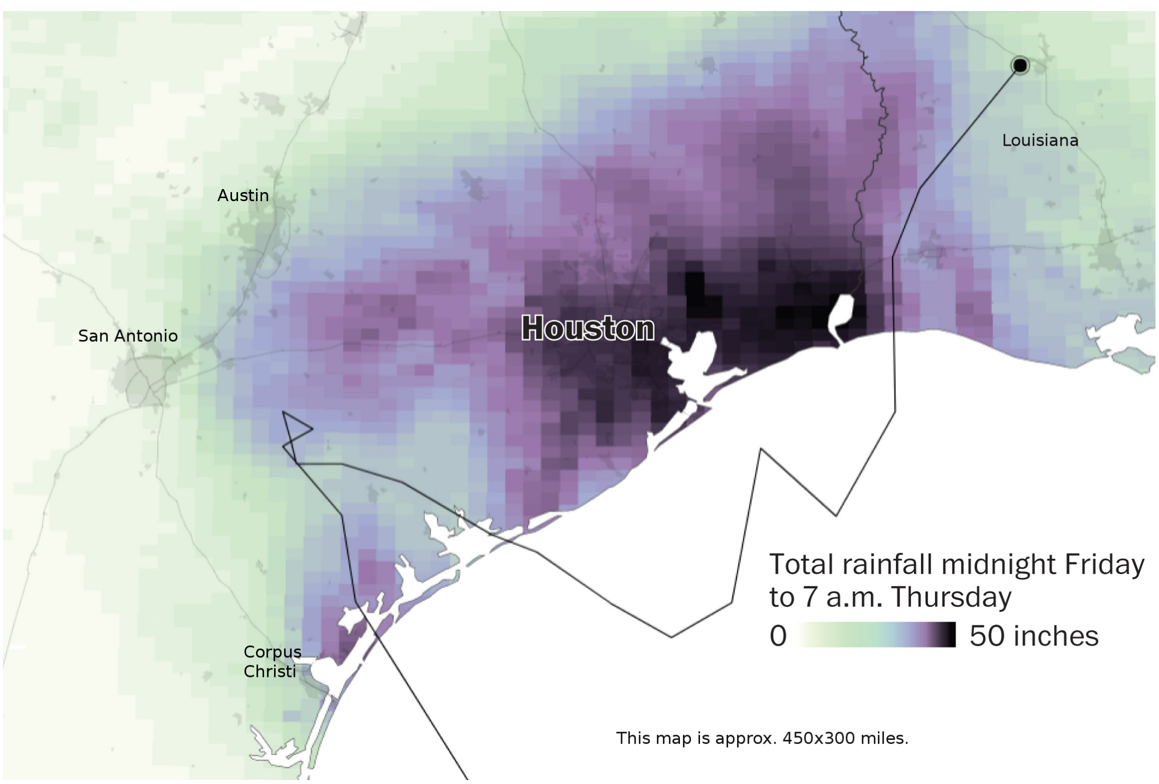


124. How many gallons of rain did Hurricane Harvey dump on the pictured land area?

   (a) Divide the 450×300 map into 50×50 squares.

   (b) Estimate rainfall at each square's midpoint, and compute a Riemann sum.

   (c) Adjust your units so that you can find the total number of gallons of water.



125. (Apex 13.1) 5,13,15,19,21

126. (Apex 13.2) 5,7,11,15,21,25

127. (Apex 13.3) 3,7,13,15

128. Find the EXACT value of a such that $\int_2^5 \left(\int_0^a \frac{x}{y} dx \right) dy = 3$.

129. Let $R = [3,8] \times [1,4]$. Use WolframAlpha to evaluate $\iint_R e^{-xy/16} dA$.

130. Let $R = [3,8] \times [1,4]$. Consider the integral $\iint_R x \sin(xy) dA$. Write the integral with $dA = dxdy$, and also write it with $dA = dydx$. Evaluate by hand whichever is easier.

131. Suppose a pond has a surface area of 3200 square feet, and an average depth of 3.64 feet. Let $d(x,y)$ be the depth at a given point in the pond's region R. Find $\iint_R d(x,y) dA$.

132. Let R be bounded by $|x - 12| = 8$ and $|y - 9| = 7$. Find the average distance of points in R to the center of that rectangle. (Do the integral in WA)

133. Let $f(x,y) = \dfrac{5x^3}{(4 + x^2 y)^{3/2}}$. Do these integrals by hand (with dx on the outside).

    (a) Do an improper integral over the horizonal strip $[0,\infty) \times [0,1]$. Does it converge?

    (b) Do an improper integral over the vertical strip $[0,1] \times [0,\infty)$. Does it converge?

    (c) Integrate over the square $[0,a] \times [0,a]$. Your answer should be a function of a.

    (d) Find the value of a that makes that last integral evaluate to 1.

134. Let $f(x,y) = xe^{(y-x^2)/5}$, and let $R = [0,a] \times [0,b]$.

    (a) Show that this function is **separable**, i.e. $f(x,y) = g(x)h(y)$.

    (b) Write $\iint_R f(x,y) dA$ as a function $V(a,b)$.

    (c) Find $\nabla V(a,b)$ where $a = 1$ and $b = 3$.

135. Evaluate $\int_0^2 \int_{y^2}^{2y} (4x - y) dxdy$ by hand

136. Sketch the region of integration, reverse the order of integration, and evaluate by hand

$$\int_0^1 \int_{2x}^2 e^{y^2} dydx$$

137. Evaluate $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy$ by hand.

138. Integrate $f(x,y) = e^x \ln y$ over the region in the 1st quadrant, above $x = \ln y$, having $|y - 3| < 2$. Do the inner integral by hand, but use W.A. for the outer integral.

139. Consider a circle of diameter D, described by $r = D\sin(\theta)$. Let P be a point on the circle's boundary. Find the average distance from a point inside the circle to the point P.