Warm-Up

1. $f(x) = x^2$ has average height of 100 over the interval $[5, b]$. Find the "exact" value of b "by hand". Follow the Ten Commandments, and include an illustrating sketch.
   Answer: $\frac{1}{b-5}\int_5^b x^2 dx = \frac{b^3-5^3}{3(b-5)} = \frac{b^2+5b+25}{3} = 100$, so $b = \frac{-5+\sqrt{1125}}{2} \approx 14.27$

Polar Coordinates

2. (Apex 9.4) 5,7,25,31,37,51

3. Convert $\left(\frac{5\pi}{12}, 20\right)_P$ to Cartesian coordinates.
   Answer: $(5.18, 19.32)_C$

4. Convert $(-4, 7)_C$ to polar coordinates.
   Answer: $(2.09, \sqrt{65})_P$

5. Convert $(\frac{\pi}{2}, -6)_C$ to polar coordinates, and use degrees to measure the angle.
   Answer: $(-75.3°, 6.20)_P$

6. List 4 aliases of $(\frac{\pi}{4}, 2)_P$.
   Answer: possible answers $(\frac{5\pi}{4}, -2)_P$, $\left(\frac{9\pi}{4}, 2\right)_P$, $\left(\frac{-3\pi}{4}, -2\right)_P$, $\left(\frac{-7\pi}{4}, 2\right)_P$

7. How can you describe all aliases of $(\frac{\pi}{3}, 7)_P$? (Hint: use expressions involving an integer k)
   Answer: $(\frac{\pi}{3} \pm 2k\pi, 7)_P$, $(\frac{4\pi}{3} + 2k\pi, -7)_P$ for integer k

8. Sketch the region described by
   (a) $r \in (2, 3]$ and $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$.
   Answer: sector shape
   (b) $|r| \leq 3$ and $|\theta| \leq \frac{\pi}{4}$
   Answer: two wedges meeting at origin

9. Find a Cartesian equation and describe the graph.
   (a) $\theta = \frac{\pi}{3}$
   Answer: $y/x = \tan(\frac{\pi}{3}) = \sqrt{3}$, so $y = \sqrt{3}x$ (line)
   (b) $r = 2\cos\theta$
   Answer: $r = 2(x/r)$, so $r^2 = 2x$, $x^2 + y^2 = 2x$, $(x^2 - 2x + 1) + y^2 = 1$, $(x-1)^2 + y^2 = 1$ (circle)
   (c) $r = \sec(\theta)\tan(\theta)$
   Answer: $r = (r/x)(y/x)$, so $y = x^2$ (parabola)

10. Find a polar equation and describe the graph.
    (a) $x^2 + y^2 = 25$
    Answer: $r = 5$ (circle)
    (b) $y = 1$
    Answer: $r = \frac{1}{\sin\theta} = \csc(\theta)$ (horizontal line)
    (c) $x + 2y = 10$
    Answer: $r = \frac{10}{\cos\theta + 2\sin\theta}$ (line with y-intercept 5 and x-int 10)

11. Show that $r = a\sin\theta + b\cos\theta$ is a circle. Find the center and radius.
    Answer: $r = a(y/r) + b(x/r)$ so $r^2 = x^2 + y^2 = ay + bx$,
    then $x^2 - bx + (b/2)^2 + y^2 - ay + (a/2)^2 = (b/2)^2 + (a/2)^2$,
    or $(x - b/2)^2 + (y - a/2)^2 = \frac{a^2+b^2}{4}$.
    So the center is $(b/2, a/2)$ and the radius is $\frac{1}{2}\sqrt{a^2 + b^2}$

12. The curve $r = \sin(\theta)\tan(\theta)$ is called a "cissoid". Prove that the graph lies inside the vertical strip $x \in [0, 1)$.
Answer: $r = (y/r)(y/x)$, so $x = \frac{y^2}{x^2+y^2}$, which is between 0 and 1.

13. Use Desmos to investigate the family of curves

$$r = \frac{1 - a\cos(\theta)}{1 + a\cos(\theta)}$$

Find transitional values of a where the curve changes its basic shape.

14. Sketch the polar curves $r = 1 + \sin\theta$ and $r = 3\sin\theta$ on the same graph, and determine where they intersect in Cartesian coordinates.
Answer: $1 + \sin\theta = 3\sin\theta$, so $1 = 2\sin\theta$, and $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$, so the graphs intersect at $(\frac{\pi}{6}, 1.5)_P$ and $(\frac{5\pi}{6}, 1.5)_P$, which are $(\pm.75\sqrt{3}, .75)_C$. Plotting (e.g. Desmos) shows that they also intersect at $(0,0)_C$ (different θ values yield $r = 0$)

15. Sketch $r = 1$ and $r = 2\cos\theta$ on the same graph. Label the intersections. Shade the region bounded between the curves.
Answer: the circles intersect at $(\pm\frac{\pi}{3}, 1)_C$ (mastercard shape)

Parametric Equations

16. (Apex 9.2) 8,11,13,17,19,25,37,39

17. Write using equations parameterized by t, and sketch the graph.

    (a) $y = e^{-x}$, $x \in [-2, 2]$
    Answer: (t, e^{-t}), $t \in [-2, 2]$

    (b) $r = e^{-\theta}$, $\theta \geq 0$
    Answer: $(e^{-t}\cos(t), e^{-t}\sin(t))$, $t \geq 0$

18. Parameterize a circle with area 154 that is centered at $(0, 12)$. For $t \in [0, \infty)$ minutes, the position should start at the top of the circle and move counter-clockwise with a frequency of 20 RPM.
    Answer: the radius is $\sqrt{154/\pi} \approx 7$, $x = -7\sin(40\pi t)$ and $y = 12 + 7\cos(40\pi t)$

19. Parameterize this ellipse for $t \geq 0$ seconds.

$$\left(\frac{x-6}{4}\right)^2 + \left(\frac{y-2}{2}\right)^2 = 1$$

    starting at $(2, 2)$, moving clockwise, with a period of 24 seconds.
    Answer: $x = 6 - 4\cos(2\pi t/24)$ and $y = 2 + 2\sin(2\pi t/24)$

20. Plot points to sketch parametric equations $(e^{2t}, t+1)$ for $t \in \mathbb{R}$. Then eliminate the parameter to write y as a function of x, and state the domain.
    Answer: $x = e^{2t}$ so $t = .5\ln(x)$, and so $y = .5\ln(x) + 1$, $x > 0$

21. Do this matching problem from the Stewart textbook:

28. Match the parametric equations with the graphs labeled I-VI. Give reasons for your choices. (Do not use a graphing device.)

(a) $x = t^4 - t + 1, \quad y = t^2$

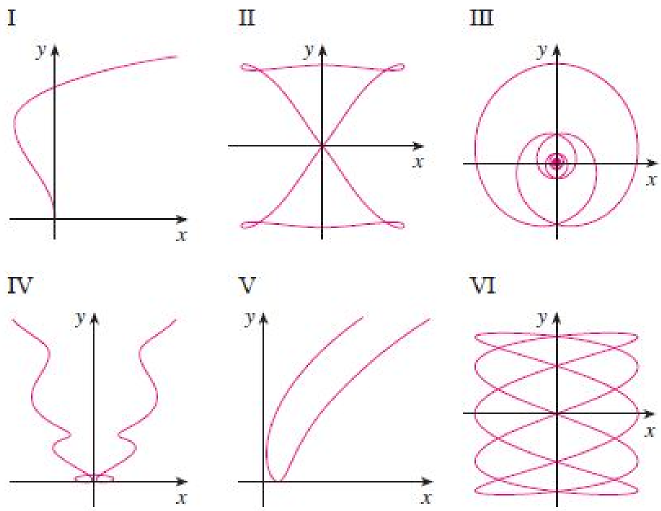
(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

(c) $x = \sin 2t, \quad y = \sin(t + \sin 2t)$

(d) $x = \cos 5t, \quad y = \sin 2t$

(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$

(f) $x = \dfrac{\sin 2t}{4 + t^2}, \quad y = \dfrac{\cos 2t}{4 + t^2}$



22. Use Desmos to investigate the "Lissajous figures" described by $x = a\sin(nt)$ and $y = b\cos(t)$ for $a, b \in \mathbb{R}$ and $n \in \mathbb{Z}$.

23. Suppose $(x(t), y(t))$ are parametric equations that describe motion that covers a path for $t \in [0, 12]$. Modify the equations to:

    (a) Shift the path 4 units north, and 7 units west.
    Answer: $(x(t) - 7, y(t) + 4)$

    (b) Shift the path 9 units at an angle of θ radians counter-clockwise from east.
    Answer: $(x(t) + 9\cos(\theta), y(t) + 9\sin(\theta))$

    (c) Move twice as fast to cover the path for $t \in [0, 6]$.
    Answer: $(x(2t), y(2t))$

    (d) Move backwards to cover the path for $t \in [0, 4]$.
    Answer: $(x(12 - 3t), y(12 - 3t))$

    (e) Double the size of the path, while maintaining its shape and starting point.
    Answer: $(2x(t) - x(0), 2y(t) - y(0))$

    (f) Bonus: rotate the path θ radians counter-clockwise around the origin.

24. Let $y = \frac{1}{1+x}$ for $x \geq 0$. The obvious parameterization is $\left(t, \frac{1}{1+t}\right)$ for $t \geq 0$. Re-parameterize in such a way that the same path is traced for $t \in [0, 1)$.
Hint: let x be a one-to-one function with domain $[0, 1)$ and range $[0, \infty)$
Answer: one possibility is $(\frac{x}{1-x}, 1 - x)$

25. Parameterize the line that goes from $(3, 15)$ to $(11, 9)$ in 5 seconds.
Answer: $x = 3 + (8/5)t$, $y = 15 - (6/5)t$

Parametric/Polar Calculus

26. (Apex 9.3) 7,15,38 (use Desmos)

27. You have parametric equations with $x(3) = 75, y(3) = 42, \frac{dx}{dt}(3) = -2, \frac{dy}{dt}(3) = 6$.

(a) Find the equation of the tangent line when $t = 3$.
Answer: $y = 42 - 3(x - 75)$

(b) Estimate $x(3.5), y(3.5)$.
Answer: $(74, 45)$

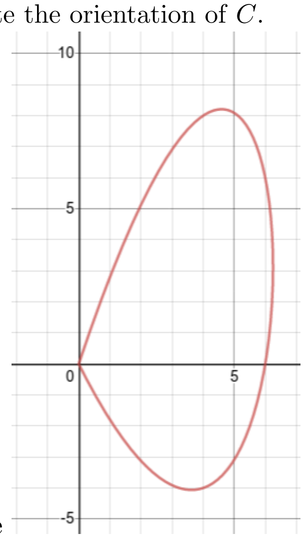
(c) Estimate $x(2.7), y(2.7)$.
Answer: $(75.6, 40.8)$

28. Consider these parametric equations that trace a curve C.

$$x = t(5 - t), \qquad y = t(5 - t)(3 - t), \qquad t \in [0, 5]$$

Let R be the region enclosed by C.

(a) Sketch the graph, and use arrows to indicate the orientation of C.



Answer: starts at origin, moves clockwise

(b) Find the equation of the tangent line when $t = 1$.
Answer: $\frac{dx}{dt} = 5 - 2t$ and $\frac{dy}{dt} = 3t^2 - 16t + 15$, so slope is $\frac{dy/dt}{dx/dt} = 2/3$ and the tan.line is $y = 8 + 2/3(x - 4)$

(c) Find the eastern-most point of R.
Answer: $t = 2.5$, $(6.25, 3.125)$

(d) Find the northern-most point of R.
Answer: $t = (16 - \sqrt{76})/6$, $(4.595, 8.209)$

(e) Find the southern-most point of R.
Answer: $t = (16 + \sqrt{76})/6$, $(3.627, -4.061)$

(f) Write the integral to find the perimeter of R, and use software to evaluate.
Answer: $\int_0^5 \sqrt{(5 - 2t)^2 + (3t^2 - 16t + 15)^2}dt = 29.18$

(g) Write the integral to evaluate the area of R, and evaluate it by hand.
Answer: $\int_0^5 y dx = \int_0^5 (t^3 - 8t^2 + 15t)(5 - 2t)dt = 625/12 \approx 52.1$

(h) What percentage of the area is above the x-axis?
Answer: $\int_0^3 y dx = 35.55$, and $35.55/52.1 = 68.2\%$

(i) Find the vertical line that splits the area of R 50-50.
Answer: look for a so that $\int_a^{5-a} y dx = .50A$, with $a \approx .9455$, we get $x = 3.83$

29. Find the tangent line to the graph described parametrically by $\left(\frac{20}{1+t^2}, 5(.8)^t\right)$ when $t = 2$.
Answer: When $t = 2$, the point is $(4, 3.2)$. Derivatives are $\frac{dx}{dt} = \frac{-40t}{(1+t^2)^2}$ and $\frac{dy}{dt} = 5(.8)^t \ln(.8)$. Evaluated at $t = 2$ gives $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \approx .223$. So the tangent line is $y = 3.2 + .223(x - 4)$

30. (Apex 9.5) 9,13,19,27,31

31. Find the length of these curves (set up integral; evaluate using software).

(a) $y = xe^{-x}$ for $x \in [0, 2]$
Answer: $\int_0^2 \sqrt{1 + (e^{-x} - xe^{-x})^2} dx \approx 2.1$

(b) $r = \sin(2\theta)$ for $\theta \in [0, \frac{\pi}{2}]$
Answer: $\int_0^{\frac{\pi}{2}} \sqrt{\sin^2(2t) + 4\cos^2(2t)} dt \approx 2.42$

(c) $(\sin(t), e^{-t^2})$ for $t \in [-\pi, \pi]$
Answer: $\int_{-\pi}^{\pi} \sqrt{\cos^2(t) + 4t^2 e^{-t^2}} dt \approx 4.9$

32. Find the area of the bounded region (set up integral; evaluate using software).

(a) $r = \sin(2\theta)$ for $\theta \in [0, \frac{\pi}{2}]$
Answer: $\int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2(2t) dt = .393$

(b) $(\sin(t), e^{-t^2})$ for $t \in [-\pi, \pi]$
Answer: $\int_{-\pi}^{\pi} e^{-t^2} \cos(t) dt \approx 1.38$

33. Find the perimeter and area of the region bounded by x-axis and the parametric curve:

$$x = (t - 1)^2$$

$$y = (t - 2)(t - 5)$$

Answer: intersects the x-axis when $y = 0$ so $t = 2, 5$, which is $x = 1, 16$
$L = (16 - 1) + \int_2^5 \sqrt{(2t - 2)^2 + (2t - 7)^2} dt \approx 31.02$
$A = |\int_2^5 (t - 2)(t - 5)2(t - 1)dt| = 22.5$

34. Consider $r = \exp(\theta)$ for $\theta \in [0, 2\pi]$.

(a) Sketch the graph.
Answer: spirals out from origin

(b) Find points where the graph is horizontal.
Answer: set $\frac{dy}{d\theta} = 0$, so $\frac{d}{d\theta} e^\theta \sin\theta = e^\theta(\sin\theta + \cos\theta) = 0$, which happens when $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

(c) Find points where the graph is vertical.
Answer: set $\frac{dx}{d\theta} = 0$, so $\frac{d}{d\theta} e^\theta \cos\theta = e^\theta(\cos\theta - \cos\theta) = 0$, which happens when $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$

35. Let $r = \sin(2\theta)$.

(a) Describe the shape of the graph.
Answer: 4 petal rose

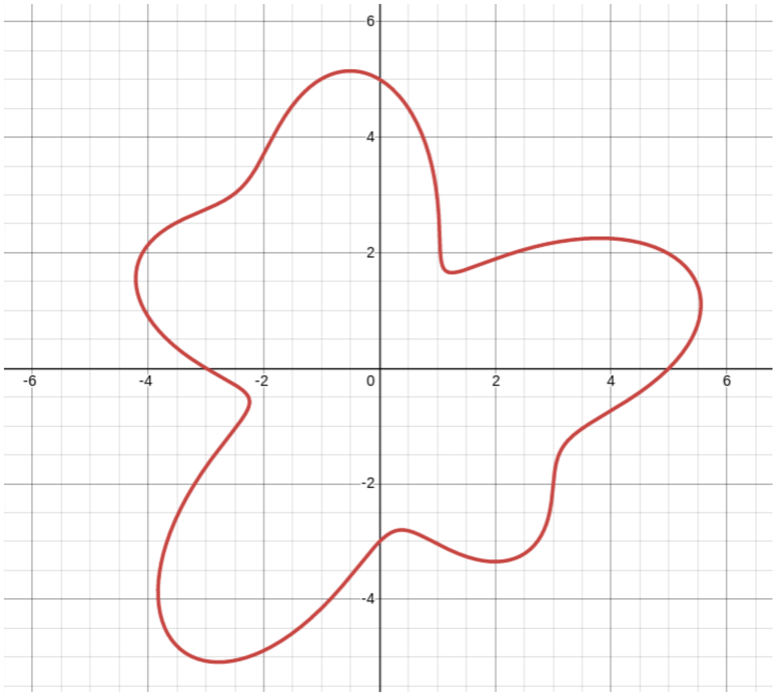
(b) Find the expression for $\frac{dy}{dx}$ in terms of θ (don't simplify).
Answer: $\frac{\frac{d}{d\theta}\sin(2\theta)\sin(\theta)}{\frac{d}{d\theta}\sin(2\theta)\cos(\theta)} = \frac{2\cos(2\theta)\sin(\theta)+\sin(2\theta)\cos(\theta)}{2\cos(2\theta)\cos(\theta)-\sin(2\theta)\sin(\theta)}$

(c) Find the slope of the graph when $\theta = \frac{\pi}{6}$.
Answer: plug in to get $\frac{dy}{dx} = 2.89$

(d) Find the equation of the tangent line at that point.
Answer: $x = 3/4$ and $y = \sqrt{3}/4 \approx .433$, so $y = .433 + 2.89(x - .75)$

36. Suppose $r(\frac{\pi}{8}) = 17$ and $\left.\frac{dr}{d\theta}\right|_{\theta=\frac{\pi}{8}} = 2.68$. Use differentials to estimate $x(\frac{\pi}{9})$ and $y(\frac{\pi}{9})$.
Answer: $r(\frac{\pi}{9}) \approx r(\frac{\pi}{8}) + 2.68(\frac{\pi}{9} - \frac{\pi}{8}) \approx 16.883$, so the Cartesian point is about $(15.865, 5.774)$

37. The zoo train track follows the polar curve $r = 4 + \cos(3\theta) + \sin(5\theta)$, pictured here.



(a) Find the length of the track (set up integral, evaluate with software).
Answer: 35.1

(b) Find the enclosed area (set up integral, evaluate with software).
Answer: 53.4

(c) Find the area north of the x-axis.
Answer: 28.3

(d) Find the area west of the y-axis.
Answer: 29.4

(e) Find the point on the track closest to the origin (hint: use calculus to minimize r).
Answer: $\frac{dr}{d\theta} = -3\sin(3\theta) + 5\cos(5\theta) = 0$, has min at $\theta \approx .97$, and $r = 2.036$

Vectors, Space, Dot, Cross

38. (Apex 10.2) 11,17,27,28,35 (see ex. 10.2.8)

39. Find a vector with length 20, that points in the opposite direction of $\vec{v} = [5, -3, 9]$.
Answer: $\frac{-20}{\sqrt{115}}\vec{v}$

40. Write the vector $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ associated with the polar point $(5, \frac{\pi}{8})_P$ in $\mathbb{R}^2$.
Answer: $\vec{v} = \begin{bmatrix} 5\cos\frac{\pi}{8} \\ 5\sin\frac{\pi}{8} \end{bmatrix}$

41. Consider points $P(1, 8, -2)$ to $Q(3, 2, 7)$ in $\mathbb{R}^3$.
    (a) find $\text{dist}(P, Q)$
    Answer: $\sqrt{2^2 + 6^2 + 9^2} = 11$
    (b) write a vector that points from P to Q
    Answer: $\vec{PQ} = [2, -6, 9]$
    (c) write a vector that points from Q to P
    Answer: $\vec{QP} = [-2, 6, -9]$
    (d) write a vector that points from P through Q, and then another 1 unit beyond
    Answer: $\vec{PQ} = \frac{12}{11}[2, -6, 9]$
    (e) write a vector that points from the origin to the midpoint of $\vec{PQ}$
    Answer: $[2, 5, 5/2]$
    (f) find the equation of the sphere centered at P with volume 100 cubic units
    Answer: $100 = \frac{4\pi}{3}r^3$, so $r \approx 2.88$, and the sphere is $(x-1)^2 + (y-8)^2 + (z+2)^2 = 8.29$
    (g) find the equation of the sphere with $\vec{PQ}$ as a diameter
    Answer: $(x-2)^2 + (y-5)^2 + (z-2.5)^2 = 5.5^2$

42. Find two unit vectors in $\mathbb{R}^2$ parallel to the tangent line of $y = e^{-x^2}$ at $x = 1$.
Answer: $\frac{dy}{dx} = -2xe^{-x^2}$, which at $x = 1$ equals $-2/e$. So the vector $[1, -2/e]$, and hence $[e, -2]$, lies parallel to the tangent line. Unit vectors are $\frac{\pm 1}{\sqrt{e^2+4}} \begin{bmatrix} e \\ -2 \end{bmatrix}$

43. True or false? Justify your answer.
$$\|\vec{a} - \vec{b}\| = \|\vec{a}\| - \|\vec{b}\|$$
Answer: false

44. This problem is from a physics book.

85 You are kidnapped by political-science majors (who are upset because you told them political science is not a real science). Although blindfolded, you can tell the speed of their car (by the whine of the engine), the time of travel (by mentally counting off seconds), and the direction of travel (by turns along the rectangular street system). From these clues, you know that you are taken along the following course: 50 km/h for 2.0 min, turn 90° to the right, 20 km/h for 4.0 min, turn 90° to the right, 20 km/h for 60 s, turn 90° to the left, 50 km/h for 60 s, turn 90° to the right, 20 km/h for 2.0 min, turn 90° to the left, 50 km/h for 30 s. At that point, (a) how far are you from your starting point, and (b) in what direction relative to your initial direction of travel are you?

45. Consider these spheres:

- $(x-3)^2 + y^2 + z^2 = 25$
- $x^2 + y^2 + z^2 = 24x - 10y + 16z - 229$

(a) Find the center and radius of each sphere.
Answer: $(3, 0, 0)$ with $r = 5$, and $(12, -5, 8)$ with $r = 2$

(b) Find the distance between the spheres.
Answer: distance between centers is $\sqrt{170}$, so subtract the radii to get $\sqrt{170} - 5 - 2 = 6.0384$

(c) Find an equation describing the set of all points equidistant from the center of each sphere.
Answer: set $(x-3)^2+y^2+z^2 = (x-12)^2+(y+5)^2+(z-8)^2$ and reduce to $18x-10y+16z = 224$

(d) Plot that equation from part (c), along with both spheres, in Geogebra3D.

46. Three points are **co-linear** if they lie in the same line. Let $P(4, 2, 7)$, Q, and R be co-linear. Find coordinates for Q and R if $\vec{PQ} = [-2, 5, 3]$ and $\vec{QR} = -1.5\vec{PR}$.
Answer: $Q = P + \vec{PQ} = (2, 7, 10)$, and $R - Q = -1.5(R - P)$ implies $2.5R = Q + 1.5P$, so $R = \frac{Q+1.5P}{2.5} = (3.2, 4, 8.2)$

47. Let $(x-6)^2 + (y-1)^2 + (z-5)^2 = 75$ be a sphere.

(a) Is the point $P(9, -4, 11)$ inside, outside, or on the sphere ?
Answer: $3^2 + 5^2 + 6^2 = 70 < 75$, so inside

(b) Find two points where the sphere intersects the x-axis.
Answer: $y = z = 0$, so $(x-6)^2 + 1 + 25 = 75$, so $(-1, 0, 0)$ and $(13, 0, 0)$

(c) Find the distance from $Q(2, 10, 8)$ to the sphere.
Answer: the distance from Q to the center is $\sqrt{16+81+9} = \sqrt{106}$; since the radius is $\sqrt{75}$, the gap is $\sqrt{106} - \sqrt{75} \approx 1.635$

48. There is a sphere S_1 with equation $(x-5)^2 + (y-7)^2 + (z-4)^2 = 9$. Find the equation of another sphere S_2 with:

- 75% more volume than S_1
- center co-linear with the origin and S_1's center
- 10 units distance between S_1 and S_2

Answer: solving $r^3 = 1.75(3^3)$ gives $r = 3.615$. The center is $\frac{\sqrt{90}+3+10+3.615}{\sqrt{90}} \begin{bmatrix} 5 \\ 7 \\ 4 \end{bmatrix} \approx [13.75, 19.25, 11]$

Then S_2 has eqn $(x - 13.75)^2 + (y - 19.25)^2 + (z - 11)^2 = 13.07$.

49. Let $P(4, 7, 3t)$ and $Q(t, 0, t^2)$ be two points parameterized by t.

   (a) Write $\text{dist}(P, Q)$ as a function of t. Graph it in Desmos.
   Answer: $\sqrt{(4-t)^2 + 49 + (3t - t^2)^2}$

   (b) Evaluate that function at time $t = 5$.
   Answer: $\sqrt{150}$

   (c) At what positive time is $\text{dist}(P, Q) = 10$?
   Answer: $t \approx 4.56$

   (d) Set $\frac{d}{dt}\text{dist}(P, Q) = 0$ to find the time at which P is closest to Q. How close do they get ?
   Answer: $t \approx 3.09$ the distance is 7.064

50. (Apex 10.3) 15,23,29,33

51. Find the angle (in degrees) between vectors $[1, 6, 4]$ and $[2, 5, 1]$.
   Answer: $\theta = \cos^{-1}(36/\sqrt{53 \cdot 30}) = 25.5°$

52. Find c so that $[1, c, 3]$ and $[-2, 2, 4]$ are orthogonal.
   Answer: $-2 + 2c + 12 = 0$, so $c = -5$

53. Let $\theta(a)$ be the angle (in degrees) between the vectors $[2, a, 7]$ and $[a^2, 1, 5]$.

   (a) Find $\theta(3)$.
   Answer: $\cos^{-1}(\frac{2a^2+a+35}{\sqrt{a^2+53}\sqrt{a^4+26}}) = \cos^{-1}(56/\sqrt{62 \cdot 107}) = 46.56°$

   (b) Find $\frac{d\theta}{da}\Big|_{a=3}$.
   Answer: look up the derivative for $\cos^{-1}$, then apply the chain rule; do a Sage demo in class; the derivative evaluates to 17.41. You can also estimate this with a difference quotient.

54. If $\|\vec{u}\| = 7$, $\|\vec{v}\| = 12$, and the sine of the angle betweeen them is 0.75, then find two possible values for $\vec{u} \cdot \vec{v}$. What is the smallest possible value of $\|\vec{u} - \vec{v}\|$?
   Answer: since $\sin(\theta) = .75$, we know $\cos(\theta) = \pm.6614$, so $\vec{u} \cdot \vec{v} = \pm 55.56$. $\|\vec{u} - \vec{v}\| = \sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})} = \sqrt{49 + 144 - 2(55.56)} = 9.05$

55. Find values of a such that the angle between $[2, 1, -1]$ and $[1, a, 0]$ is $45°$.
   Answer: setting $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\| \cos\theta$, gives $2 + a = \sqrt{6}\sqrt{1 + a^2}\frac{\sqrt{2}}{2}$. Solving, we get $a = 1 \pm \sqrt{6}/2$.

56. Find the acute angle between $y = x^2$ and $y = x^3$ where they intersect at $(1, 1)$.
   Answer: the slopes are 2 and 3 respectively, which can be represented by vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. The angle between them is $\cos^{-1}(7/\sqrt{5}/\sqrt{10}) \approx .142$, or $8.13°$.

57. $\|\vec{PQ}\| = 19$, $\|\vec{RQ}\| = 26$, and $\vec{PQ} \cdot \vec{RQ} = 494$. Explain how you know P, Q, R are co-linear. Find $\text{dist}(P, R)$.
   Answer: $\cos(\theta) = 1$, so the angle is zero. But sketch a diagram to see that if PQ and RQ point in the same direction, then $\text{dist}(P, R) = 26 - 19 = 7$.

58. Find the acute angle (in degrees) between $\vec{v} = [1; 1; 1]$ and

   (a) the y-axis
   Answer: using $[0, 1, 0]$ as the other vector, the angle is $\cos^{-1}(1/\sqrt{3}) \approx 54.7$

   (b) the xy-plane
   Answer: using $[1, 1, 0]$ as the other vector, the angle is $\cos^{-1}(2/\sqrt{6}) \approx 35.3$

59. Find the unit vectors in $\mathbb{R}^2$ that make a 20° angle with the vector $[1, 7]$.
Answer: Write the unit vector as $\begin{bmatrix} c \\ s \end{bmatrix}$ where $c^2 + s^2 = 1$. Then solve $\begin{bmatrix} c \\ s \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 7 \end{bmatrix} = \sqrt{50}\cos(\pi/9)$, which can be solved with W.Alpha to get $\begin{bmatrix} .471 \\ .882 \end{bmatrix}$ or $\begin{bmatrix} -.206 \\ .979 \end{bmatrix}$.
Another method: the angle of the desired vectors are $\tan^{-1}(7) \pm \frac{\pi}{9} = 1.7779, 1.0798$. Then the cosine/sine components give the answers.

60. Show that if $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ are orthogonal, then the vectors $\vec{u}$ and $\vec{v}$ have the same length.
Answer: we know $0 = (u + v) \cdot (u - v) = u \cdot u - v \cdot v$, so $\|u\|^2 = \|v\|^2$, showing they have the same magnitude

61. (Apex 10.4) 19,23,31,35

62. Consider points $P(7, 0)$, $Q(0, 5)$ and $R(a, a)$.

   (a) Sketch the triangles for $a = 2$ and $a = 9$ on the same graph.
   (b) Find the perimeter of $\triangle PQR$ as a function $P(a)$.
   Answer: $P(a) = \sqrt{74} + \sqrt{a^2 + (a - 5)^2} + \sqrt{(a - 7)^2 + a^2}$
   (c) Find the area of $\triangle PQR$ as a function $A(a)$.
   Answer: $A(a) = \frac{1}{2}\|\vec{PQ} \times \vec{PR}\| = \frac{1}{2}|7a + 5(a - 7)|$
   (d) Find $\frac{dA}{da}$
   Answer: 12
   (e) For what value of a is the area zero ?
   Answer: $a = 35/12$
   (f) Embed $P(7, 0, 0)$ and $Q(0, 5, 0)$ into $\mathbb{R}^3$, and raise the third point to $R(a, a, a)$. Now find the value of a that minimizes the area of $\triangle PQR$.
   Answer: $2A^2 = (5a)^2 + (7a)^2 + (12a - 35)^2$ is minimized if $a = 840/436$

63. If $\vec{v} \cdot \vec{w} = 6$ and $\|\vec{v} \times \vec{w}\| = 11$, find the angle (in degrees) between $\vec{v}$ and $\vec{w}$.
Answer: $\tan\theta = \frac{11}{6}$ so $\theta = 61.4°$

64. Consider points $P(1, 3, 5)$, $Q(2, 4, 8)$, $R(4, 9, 11)$, and $S(x, 7, 2)$.

   (a) Let $x = 6$ and find the volume of the parallelpiped with adjacent corners at P, Q, R, S.
   Answer: 57
   (b) Find the value of x so that S is co-planar with ΔPQR.
   Answer: Set the triple scalar product to zero. $\vec{PS} \cdot (\vec{PQ} \times \vec{PR}) = \begin{bmatrix} x - 1 \\ 4 \\ -3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \right) = \begin{bmatrix} x - 1 \\ 4 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -12 \\ 3 \\ 3 \end{bmatrix}$, and solve $-12(x - 1) + 12 - 9 = 0$ to get $x = 1.25$

65. A parallelpiped determined by adjacent corners: $P(0, 0, 0)$, $Q(3, 1, 2)$, $R(-1, 4, 1)$, $S(2, 5, z)$, has volume 100.

   (a) If $z > 0$, find its value.
   Answer: set $100 = \vec{PS} \cdot (\vec{PQ} \times \vec{PR}) = \begin{bmatrix} 2 \\ 5 \\ z \end{bmatrix} \cdot \begin{bmatrix} -7 \\ -5 \\ 13 \end{bmatrix} = -39 + 13z$ to get $z = \frac{139}{13}$

(b) If $z < 0$, find its value.

Answer: set $100 = \vec{PS} \cdot (\vec{PR} \times \vec{PQ}) = \begin{bmatrix} 2 \\ 5 \\ z \end{bmatrix} \cdot \begin{bmatrix} 7 \\ 5 \\ -13 \end{bmatrix} = 39 - 13z$ to get $z = \frac{-61}{13}$

66. A molecule of methane, CH_4, is structured with four hydrogen atoms at the vertices of a regular tetrahedron and the carbon atom at the centroid. The bond angle is the angle formed by $H - C - H$ combination; it is the angle between the lines that join the carbon atom to two of the hydrogen atoms. Show that the bond angle is about $109.5°$.

    Hint: Take the vertices of the tetrahedron to be the points $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, and $(1, 1, 1)$. Then the centroid is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

    Answer: angle between $(\frac{1}{2} - 1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2} - 1, \frac{1}{2})$ has $\cos\theta = \frac{-1/4}{\sqrt{9/16}} = -1/3$, and $\cos^{-1}(-1/3) =$ 109.47 degrees.

67. The Gordon line ($y = x$) intersects the curve $y = \sqrt{x}$ at $(0, 0)$ and $(1, 1)$. Find the acute angle (in degrees) of intersection at each of those points.
    Answer: at $(0, 0)$, $y = \sqrt{x}$ is vertical, so the angle is $45°$.
    at $(1, 1)$, slopes are 1 and 1/2 respectively, so we want the angle between vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\.5\end{bmatrix}$ which is $\cos^{-1}(1.5/\sqrt{2 * 1.25}) = 18.4°$

Lines, Planes

68. (Apex 10.5) 9,11,17

69. (Apex 10.6) 9,11,13,19,21,25

70. Find the equation of the plane through $(5, 8, 3)$ and orthogonal to the vector $[7, -2, 4]$.
    Answer: $7(x - 5) - 2(y - 8) + 4(z - 3) = 0$

71. A sphere is centered at $(7, -2, 15)$ and the vector $\vec{v} = [9, 13, 5]$ goes from the center to the surface of the sphere.

    (a) Find the equation of the sphere.
    Answer: $(x - 7)^2 + (y + 2)^2 + (z - 15)^2 = 275$

    (b) Find the equation of the axial line through the sphere's center in the direction of $\vec{v}$.
    Answer: $x = 7 + 9t$, $y = -2 + 13t$, $z = 15 + 5t$

    (c) Find the equation of the plane, normal to $\vec{v}$, that splits the sphere into two hemispheres.
    Answer: $9(x - 7) + 13(y + 2) + 5(z - 15) = 0$

72. Let $P(2, 4, 9)$, $Q(7, 3, 5)$, and $R(1, 8, 2)$ be three points in $\mathbb{R}^3$.

    (a) Find the equation of the plane containing those points.

    (b) Find the equation of the line through P that is perpendicular to the plane.

    (c) Find the area of $\triangle PQR$.

73. Consider the line $\vec{\ell}(t) = \begin{bmatrix}3 + 5t\\1 - 7t\\8 + 2t\end{bmatrix}$ and the plane $z - 15 = 2(x - 3) + 9(y + 1)$.

    (a) Find the point of intersection.
    Answer: $8 + 2t - 15 = 2(5t) + 9(2 - 7t)$ is true for $t = 5/11$, so the point is $(5.27, -2.18, 8.91)$

    (b) Find the acute angle (in degrees) of intersection.
    Answer: the plane's normal vector is $[2, 9, -1]$ which makes an angle of $\cos^{-1}(-55/\sqrt{6708}) =$ $132.2°$ with the line. Subtracting 90, the angle with the plane itself is $42.2°$.

74. Consider the line $\vec{\ell}(t) = \begin{bmatrix} 3+5t \\ 1-7t \\ 8+2t \end{bmatrix}$ and the sphere $(x-3)^2 + (y+1)^2 + (z-15)^2 = r^2$.

(a) Find the values of t, and the correspondings point of intersection if $r = 10$.
Answer: $t = -.496, 1.214$ gives $(.519, 4.47, 7.01)$ and $(9.07, -7.5, 10.43)$

(b) Find the EXACT value of r such that line is tangent to the sphere (one intersection). Then find that EXACT point of tangency.
Answer: working it out to have one solution with the QF, you get $56^2 - 4(78)(53 - r^2) = 0$ so $r = \sqrt{13400}/312$, and $t = 14/39$ and the point $(187/39, -59/39, 340/39)$

(c) Using a computer, find the value of r such that the points of intersection are 50 units apart.
Answer: using e.g. Sage, $r = 25.845$

75. These two lines are known to intersect. What is the value of c ? Follow these steps:

$$\vec{\ell}_1(t) = \begin{bmatrix} 4t \\ 2t-7 \\ 3t-2 \end{bmatrix} \qquad \vec{\ell}_2(t) = \begin{bmatrix} 14-6t \\ 12+t \\ 5+ct \end{bmatrix}$$

(a) Using different symbols for the parameters, e.g. t_1 and t_2, set the x and y coordinates equal and find the point of intersection. (you are ignoring the z-coordinates for now)
Answer: $t_1 = 8$ and $t_2 = -3$ give $(32, 9, 22)$

(b) Now set the z coordinates equal to find the value of c.
Answer: $22 = 5 - 3c$ gives $c = -17/3$

76. Find the line of intersection between these planes. Follow the steps:

$$z + 80 = 2(x-3) + 5(y+1) \qquad 4x + y + 3z = 45$$

(a) Set $x = 0$ and solve for y and z to find one point on the line.
Answer: $z + 81 = 5y$ and $y + 3z = 45$ have solution $(0, 18, 9)$

(b) Set $z = 0$ and solve for x and y to find one point on the line.
Answer: $2x + 5y = 81$ and $4x + y = 45$ have solution $(8, 13, 0)$

(c) Write parametric equations of the line through those two points.
Answer: $x = 8t$, $y = 18 - 5t$, $z = 9 - 9t$

77. Find the value of z so that the point $(8, 2, z)$ is on the plane $3(x-8) + 5(y-2) + 9(z-1) = 36$.
Answer: solve $9(z-1) = 36$ to get $z = 5$

78. Find an equation for the set of all points equidistant from the points $A(-1, 5, 3)$ and $B(6, 2, -2)$.
Answer: reduce $(x+1)^2 + (y-5)^2 + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2$ to get the plane $14x - 6y - 10z = 9$

79. Find a vector $\vec{v}$ that satisfies these conditions:

- $\|\vec{v}\| = 12$
- $\vec{v} \cdot \vec{k} = 1$
- parallel to the plane $3x - 5y + 7z = 74$

Answer: let $\vec{v} = [a, b, 1]$, then $a^2 + b^2 + 1 = 144$, and $3a - 5b + 7 = 0$ (since $\vec{v}$ is orthogonal to the plane's normal vector). Solve for a, b. This problem can be simplified by relaxing the first two conditions.

80. Consider the plane with x, y, and z intercepts of 24, 6, and 15 respectively.

(a) Find the equation of the plane.
Answer: $90x + 360y + 144z = 2160$

(b) Find the equation of the line through the origin and the "centroid" of the triangle of intercepts.
Answer: $\ell(t) = [8t, 2t, 5t]$

(c) Find the angle (in degrees) at which that line pierces the plane.
Answer: angle between $[90, 360, 144]$ and $[8, 2, 5]$ is $55.76°$, so the angle between the line and plane is $|90 - 55.76| = 34.24°$

Multi-variable Functions

81. (Apex 12.1) 11,13,19,21

82. (Apex 12.3) 5,9,11,15,17,27,29,31

83. Let $z = x^2 - y$. The contour that contains (3,2) also passes through $(8, y)$ for what value of y?
Answer: $z(3,2) = 7$, so at $(8, y)$ we know $7 = 8^2 - y$, so $y = 57$

84. Let $z = f(x,y) = \exp(-x^2 - y^2)$.

    (a) Find the range.
    Answer: $(0, 1]$

    (b) Find the radius of the circular contours for $z = .5$, $z = .2$, and $z = .1$.
    Answer: $\ln(.5) = -x^2 - y^2$, so $x^2 + y^2 = \ln(2)$, so $r = \sqrt{\ln(2)} \approx .8325$
    $\ln(.2) = -x^2 - y^2$, so $x^2 + y^2 = \ln(5)$, so $r = \sqrt{\ln(5)} \approx 1.269$
    $\ln(.1) = -x^2 - y^2$, so $x^2 + y^2 = \ln(10)$, so $r = \sqrt{\ln(10)} \approx 1.517$

85. Let $z = f(x,y) = \log_2(xy + 1)$.

    (a) Find the domain and shade it on the xy-plane.
    Answer: $xy + 1 > 0$ is true on the region between the two branches of the hyperbola $y = \frac{-1}{x}$

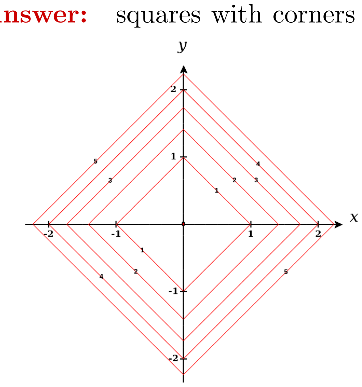
    (b) Sketch the contours corresponding to $z = -2$, $z = -1$, $z = 0$, $z = 1$, and $z = 2$.
    Answer:
    $-2 = \log_2(xy + 1)$ so $.25 = xy + 1$ and $y = -.75/x$
    $-1 = \log_2(xy + 1)$ so $.5 = xy + 1$ and $y = -.5/x$
    $0 = \log_2(xy + 1)$ so $1 = xy + 1$ and $xy = 0$, which are the x and y axes
    $1 = \log_2(xy + 1)$ so $2 = xy + 1$ and $y = 1/x$
    $2 = \log_2(xy + 1)$ so $4 = xy + 1$ and $y = 3/x$

86. Contrast the contours of $z = 2x + y$ and $z = \sin(2x + y)$.
Answer: In both cases the contours are lines of the form $y = -2x + c$. In the former, the contours are equally spaced. In the latter, the lines are closer together when $2x + y$ is a multiple of π

87. Let $f(x,y) = |x| + |y|$. For this problem, consider contours for $z = 0, 1, 2, 3, 4, 5$

    (a) Sketch the contours of $z = f(x,y)$. Describe the shape and spacing.
    Answer: squares with corners on the axes. equally spaced

    (b) Sketch the contours of $z = f^2(x,y) = (|x| + |y|)^2$. Describe the shape and spacing.
    Answer: squares with corners on the axes. closer together as you move away from the origin



(c) Sketch the contours of $z = \sqrt{f(x, y)} = \sqrt{|x| + |y|}$. Describe the shape and spacing.
Answer: squares with corners on the axes. further apart as you move away from the origin

88. (Apex 12.4) 5,9,19,21

89. (Apex 12.6) 7,13,19

90. (Apex 12.7) 9,13,17,23

91. Here is a table of points for a function $z = f(x, y)$.

x	y	z
7.00	3.85	14.50
7.03	4.00	14.90
6.97	4.00	15.40
7.00	4.15	16.10

(a) Average the surrounding points to estimate $f(7, 4)$.
Answer: 15.225

(b) Use difference quotients to estimate $\nabla f(7, 4)$.
Answer: $fx \approx \frac{14.9-15.4}{7.03-6.97} = 8.33$ and $f_y \approx \frac{16.1-14.5}{4.15-3.85} = 2$

(c) Use those answers to write an approximate tangent plane at $(7, 4)$.
Answer: $z = 15.225 + 8.33(x - 7) + 2(y - 4)$

(d) Estimate $f(7.05, 4.09)$.
Answer: $15.225 + 8.33(.05) + 2(.09) = 15.8215$

92. If $dz = 2dx + 5dy + 4dt$, find the equation of the tangent plane at $(7, 0, 3, 14)$.
Answer: $z = 14 + 2(x - 7) + 5y + 4(t - 4)$

93. You have a company that sells snowshovels. Quantity sold (S) is a function of price (P dollars), temperature (T degrees), and advertising (A units). Suppose $dS = -25dP - 7dT + 4dA$.

(a) Find $\frac{\partial S}{\partial P}$, and write a sentence that explains its sign.
Answer: $\frac{\partial S}{\partial P} = -25$ is negative since as you raise the price, fewer will be sold.

(b) Find $\frac{\partial S}{\partial T}$, and write a sentence that explains its sign.
Answer: $\frac{\partial S}{\partial T} = -7$ is negative since as temperature rises, presumably less snow, so fewer shovels needed.

(c) Find $\frac{\partial S}{\partial A}$, and write a sentence that explains its sign.
Answer: $\frac{\partial S}{\partial A} = 4$ is positive since if you advertise, you should attract some extra customers.

(d) If temperature falls 10 degrees, you raise the price by a dollar, and reduce advertising by 6 units, estimate the net affect on S.
Answer: $dS = -25(1) - 7(-10) + 4(-6) = 21$

94. A bird is flying south for the winter. Let $T(t, x, y)$ be the expected daily high temperature as a function of time in days, degrees east longitude, and degrees north latitude. At the current time and location, suppose $T = 62$ and $\nabla T = [-0.18, .07, -1.53]$. Over the next week, the bird hopes to move 2° east and 5° south. Find the differential dT.
Answer: $dT = -.18(7) + .07(2) + (-1.53)(-5) = 6.53$

95. Let $f(x, y) = \frac{5x}{1+y^2}$.

(a) Evaluate $f(9, 3)$.
Answer: 4.5

(b) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ as functions of x and y.
Answer: $\frac{\partial f}{\partial x} = \frac{5}{1+y^2}$ and $\frac{\partial f}{\partial y} = \frac{-10xy}{(1+y^2)^2}$

(c) Evaluate the gradient $\nabla f(9, 3)$.
Answer: $[.5, -2.7]$

(d) Find the equation of the tangent plane at $(9, 3)$.
Answer: $z = 4.5 + .5(x - 9) - 2.7(y - 3)$

(e) Find the equation of the normal line.
Answer: $x = 9 + .5t, y = 3 - 2.7t, z = 4.5 - t$

(f) Using the linearization (i.e. tangent plane) estimate $f(9.17, 2.88)$.
Answer: $4.5 + .5(.17) - 2.7(-.12) = 4.909$

(g) Evaluate $f(9.17, 2.88)$ exactly using the non-linear formula for f.
Answer: 4.933

(h) Find the directional derivatives $D_{\vec{v}} f(9, 3)$ for these directions:

i. $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
Answer: .5

ii. $\vec{v} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$
Answer: .5

iii. $\vec{v} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$
Answer: -.5

iv. $\vec{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$
Answer: 2.7

v. $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$
Answer: 2.46

vi. $\vec{v} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$
Answer: -2.3

vii. $\vec{v} = \begin{bmatrix} -5 \\ -12 \end{bmatrix}$
Answer: 2.3

viii. $\vec{v} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$
Answer: .937

96. Consider a multivariable function with $f(29, 52) = 73$ and $\nabla f(29, 52) = \begin{bmatrix} -0.23 \\ 0.09 \end{bmatrix}$.

(a) Find the equation of the tangent plane at the given point.
Answer: $z = 73 - .23(x - 29) + .09(y - 52)$

(b) Estimate $f(29.5, 50.2)$
Answer: $73 - .23(.5) + .09(-1.8) = 72.723$

(c) Find the equation of the normal line.
Answer: $\vec{r}(t) = \begin{bmatrix} 29 - .23t \\ 52 + .09t \\ 73 - t \end{bmatrix}$

(d) Let $\vec{v} = [a; 7]$. If $a = -2$, find $D_{\vec{v}} f(29, 52)$.
Answer: $\frac{\nabla f \cdot [-2;7]}{\sqrt{2^2+7^2}} = .1497$

(e) Find the value of a such that $D_{\vec{v}} f(29, 52) = 0$
Answer: solve $-.23a + .09(7) = 0$ to get $a = 63/23 \approx 2.74$

97. Suppose at a certain point, $\nabla f = \begin{bmatrix} 0.21 \\ -0.56 \end{bmatrix}$ and the Hessian is $H = \begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}$. Consider the velocity direction $\vec{v} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

(a) Find the slope: $D_{\vec{v}} f$.
Answer: $-3.29/\sqrt{58} = -.432$

(b) Find the concavity: $D_{\vec{v}}(D_{\vec{v}} f)$.
Answer: $\vec{v}^T H \vec{v} / (v \cdot v) = 4.45$

(c) If you move in the opposite direction, $-\vec{v}$, then what are the slope and concavity ?
Answer: slope is .432 and concavity is 4.45

(d) Find the steepest ascent direction, and the slope in that direction.
Answer: $[.21, -.56]$ slope is .598

(e) Find the steepest descent direction, and the slope in that direction.
Answer: $[-.21, .56]$ slope is -.598

(f) Find a direction where the slope is zero.
Answer: $\perp$ to gradient, e.g. $[.56, .21]$

(g) Find a direction where the slope is 0.35.
Answer: with $\vec{v} = \begin{bmatrix} c \\ s \end{bmatrix}$ with $c^2 + s^2 = 1$, solve $.21c - .56s = .35$ (WA) to get two possibilities: $\vec{v} = \begin{bmatrix} -.554 \\ -.833 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} .965 \\ -.263 \end{bmatrix}$

(h) Find a direction where the slope is -0.35.
Answer: $\vec{v} = \begin{bmatrix} .554 \\ .833 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -.965 \\ .263 \end{bmatrix}$

98. Let $f(x, y) = x^3 + \frac{15x}{y}$

(a) Find the gradient $\nabla f(x, y)$; then evaluate $\nabla f(2, 5)$.
Answer: $\nabla f(x, y) = [3x^2 + 15/y, -15x/y^2]$, $\nabla f(2, 5) = [15, -1.2]$

(b) Find the Hessian matrix $H(x, y)$; then evaluate $H(2, 5)$.
Answer: $H(x, y) = \begin{bmatrix} 6x & -15/y^2 \\ -15/y^2 & 30x/y^3 \end{bmatrix}$, $H(2, 5) = \begin{bmatrix} 12 & -.6 \\ -.6 & .48 \end{bmatrix}$

(c) Find the equation of the tangent plane at $(2, 5)$.
Answer: $z = 14 + 15(x - 2) - 1.2(y - 5)$

(d) Find the equation of the normal line at $(2, 5)$.
Answer: $\ell(t) = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix} + \begin{bmatrix} 15 \\ -1.2 \\ -1 \end{bmatrix} t$

99. This plane lies tangent to the surface of $f(x, y)$.

$$z = 425 + 3(x - 90) + 7(y + 260)$$

(a) Find the evident point P of tangency.
Answer: $(90, -260, 425)$

(b) Find ∇f at P.
Answer: $\nabla f = [3; 7]$

(c) Find the slope of the surface in the steepest ascent direction.
Answer: $\|\nabla f\| = \sqrt{58}$

(d) Estimate $f(86, -257)$.
Answer: $425 + 3(-4) + 7(3) = 434$

(e) Find the equation of the line through P orthogonal to the surface of f.
Answer: $\ell(t) = \begin{bmatrix} 90 \\ -260 \\ 425 \end{bmatrix} + \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} t$

(f) Find the equation of the tangent line to the contour $f(x, y) = 425$.
Answer: the contour is $\perp$ to the gradient vector, so a direction is $[-7, 3]$. $\ell(t) = \begin{bmatrix} 90 \\ -260 \end{bmatrix} + \begin{bmatrix} -7 \\ 3 \end{bmatrix} t$, or $y = -260 - \frac{3}{7}(x - 90)$

100. Suppose at a certain point, the surface of a function has

- slope 7 in the direction $[3, 4]$
- slope 1 in the direction $[8, 15]$

Find ∇f.

Answer: solve the linear system $(3f_x + 4f_y)/5 = 7$ and $(8f_x + 15f_y)/17 = 1$ to get $\nabla f = \begin{bmatrix} 35.154 \\ -17.615 \end{bmatrix}$

101. Suppose $f(200, 700) = 125$, $f(201, 704) = 143$, and $f(198, 703) = 119$.

(a) Estimate $\nabla f(200, 700)$.

Hint: $f(200 + dx, 700 + dy) - f(200, 700) \approx \nabla f(200, 700) \cdot \begin{bmatrix} dx \\ dy \end{bmatrix}$

Answer: $18 \approx f_x + 4f_y$ and $-6 \approx -2f_x + 3f_y$. Solve this linear system to get $f_x \approx 7.09$ and $f_y \approx 2.73$

(b) Estimate $f(203, 695)$.

Answer: $125 + 3(7.09) - 5(2.73) = 132.6$

102. Suppose you are on a hill and the steepest descent direction is east by southeast (i.e. $\theta = -11.25°$), having slope -0.35. Find the slope in the southwest direction ($\theta = 225°$).

Answer: $\nabla f = .35 \begin{bmatrix} \cos(168.75) \\ \sin(168.75) \end{bmatrix} \approx \begin{bmatrix} -.343 \\ .0683 \end{bmatrix}$, so the southwest slope is $\frac{1}{\sqrt{2}} \begin{bmatrix} -.343 \\ .0683 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} \approx .1945$

another method: the slope is $\|\nabla f\| \cos(\theta) = .35 \cos(56.25°) = .1945$

103. Consider the surface of $f(x, y) = xy$ and the line $\ell(t) = \begin{bmatrix} 1 + 2t \\ 2 + 3t \\ 146 - 23t \end{bmatrix}$.

(a) Find the points of intersection.

Answer: solve $(1 + 2t)(2 + 3t) = 146 - 23t$ to get $t = 3, -8$, which are points $(7, 11, 77)$ and $(-15, -22, 330)$

(b) Find the acute angles (in degrees) at which the line intersects the surface at those points.

Answer: $\nabla f = [y, x]$,
so at the first point $\vec{n} = [11, 7, -1]$, and the angle is $90 - \cos^{-1}(66/\sqrt{542}/\sqrt{171}) \approx 12.52$
at the other point $\vec{n} = [-22, -15, -1]$ and the angle is $90 - \cos^{-1}(66/\sqrt{542}/\sqrt{710}) \approx 6.11$

104. Let $f(x, y) = \frac{x^2}{y}$.

(a) Find the equation of the tangent plane at $(6, 4)$.

Answer: $f(6, 4) = 9$ and $\nabla f = [2x/y, -x^2/y^2] = [3, -2.25]$, so $z = 9 + 3(x - 6) - 2.25(y - 4)$

(b) Find the slope of the surface in the direction of $\vec{v} = [8, -5]$.

i. At an arbitrary point (x, y).

Answer: $D_{\vec{v}} f = [2x/y, -x^2/y^2] \cdot [8, -5]/\sqrt{89} = \frac{1}{\sqrt{89}}(16x/y + 5x^2/y^2)$

ii. At $(6, 4)$.

Answer: $D_{\vec{v}} f(6, 4) = [3, -2.25] \cdot [8, -5]/\sqrt{89} \approx 3.74$

(c) Find the concavity in the direction of $\vec{v}$.

i. At an arbitrary point (x, y).

Answer: $D_{\vec{v}}(D_{\vec{v}} f) = \frac{1}{89} \begin{bmatrix} 16/y + 10x/y^2 \\ -16x/y^2 - 10x^2/y^3 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ -5 \end{bmatrix} = \frac{1}{89}(128/y + 160x/y^2 + 50x^2/y^3)$

ii. At $(6, 4)$.

Answer: plug in to get ≈ 1.35

105. Find all points at which the direction of fastest change of $f(x, y) = x^2 + y^2 - 2x - 4y$ is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Answer: set $\nabla f = \begin{bmatrix} 2x-2 \\ 2y-4 \end{bmatrix} = c \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, so $3(2x-2) = 2(2y-4)$, or $y = \frac{1}{2}(1+3x)$

106. Show that $v = (x - at)^4 + \cos(x + at)$ satisfies the wave equation:

$$\frac{\partial^2 v}{\partial t^2} = a^2 \frac{\partial^2 v}{\partial x^2}$$

107. Show that $f(x, y) = \tan^{-1}\left(\frac{y}{x}\right)$ satisfies the Laplace equation:

$$f_{xx} + f_{yy} = 0$$

108. Let $f(x, y) = \ln(10 + x^2 + 3x + y^2)$

(a) Find $\nabla f(x, y)$, then evaluate $\nabla f(2, 5)$.
Answer: $\frac{1}{10+x^2+3x+y^2} \begin{bmatrix} 2x+3 \\ 2y \end{bmatrix}$ evaluates to $\begin{bmatrix} 7/45 \\ 10/45 \end{bmatrix}$

(b) Find the Hessian matrix $H(x, y)$.
Answer: For convenience, let $\gamma = 10 + x^2 + 3x + y^2$.
Then $H(x, y) = \frac{1}{\gamma^2} \begin{bmatrix} 2\gamma - (2x+3)^2 & -2y(2x+3) \\ -2y(2x+3) & 2\gamma - 4y^2 \end{bmatrix}$ evaluates to $\frac{1}{\gamma^2} \begin{bmatrix} 41 & -70 \\ -70 & -10 \end{bmatrix}$

(c) Let $(2, 5)$ be your current location, and consider walking in the walking in the 11:20 clock direction.

i. Write a unit vector representing your direction (use cos/sin of your angle).
Answer: $\vec{v} = \begin{bmatrix} \cos(110) \\ \sin(110) \end{bmatrix} = \begin{bmatrix} -.342 \\ .9397 \end{bmatrix}$

ii. Find the slope.
Answer: $D_{\vec{v}} f(a, b) = \nabla f \cdot \vec{v} = .1556$

iii. Find the concavity using the formula $\vec{v} \cdot (H\vec{v})$.
Answer: .020226

Chain Rule

109. (Apex 12.5) 11,17,21,29

110. Let z be a function of x and y, and let both x and y be functions of s and t. At a certain point,

$$\nabla z = \begin{bmatrix} 6 \\ -1 \end{bmatrix} \qquad \nabla x = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \qquad \nabla y = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.
Answer: $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (6)(4) + (-1)(7) = 17$
$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (6)(9) + (-1)(3) = 51$

111. Let $P = \sqrt{u^2 + v^2 + w^2}$, where $u = xe^{-y}$, $v = ye^x$, and $w = \frac{xy}{e^x}$. Find $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ using the chain rule.
Answer: Notice that e.g. $\frac{\partial P}{\partial u} = \frac{1}{2}(u^2 + v^2 + w^2)^{-1/2}(2u) = \frac{u}{P}$. Then:
$\frac{\partial P}{\partial x} = \frac{u}{P}(e^{-y}) + \frac{v}{P}(ye^x) + \frac{w}{P}\frac{ye^x - xye^x}{e^{2x}}$
$\frac{\partial P}{\partial y} = \frac{u}{P}(-xe^{-y}) + \frac{v}{P}(e^x) + \frac{w}{P}(xe^{-x})$

112. A right circular cone is being expanded by animation software. It currently has sliders set for radius 25 and height 60 cm. Let r be the radius, h the height, and V the volume.

(a) If height is increasing at 5 cm/sec, but the radius is holding constant, find $\frac{dV}{dt}$.
Answer: use the formula $V = \frac{\pi}{3}r^2h$ and the chain rule: $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$

(b) If height is increasing at 5 cm/sec, but the radius is decreasing by 2 cm/sec, find $\frac{dV}{dt}$.

(c) If height is increasing at 5 cm/sec, and volume is holding constant, find $\frac{dr}{dt}$.

(d) If height is increasing at 5 cm/sec, and volume is increasing by 40 cm^3/sec , find $\frac{dr}{dt}$.

113. There is a circular (as viewed from above) path centered at the origin with radius 800 ft. Let $z = f(x, y)$ be the ground's elevation at a particular point, and suppose

$$\nabla f(x, y) = 10^{-11}\begin{bmatrix} 3x^2y + 10xy^2 \\ x^3 + 10yx^2 \end{bmatrix}$$

You observe someone start walking from the southernmost point on the path, going clockwise at 5 ft/sec (apparently from above). The walker's elevation undulates with the surface of the ground.

(a) Write parametric equations for the x and y coordinates of location, for $t \in [0, \infty)$.
Answer: note that the period is $1600\pi/5 = 320\pi$ seconds.
$x = -800\sin(t/160)$ and $y = -800\cos(t/160)$

(b) Find $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
Answer: $\frac{dx}{dt} = -5\cos(t/160)$ and $\frac{dy}{dt} = 5\sin(t/160)$

(c) Use the multi-variable chain rule to find $\frac{dz}{dt}$ when $t = 180$ seconds.
(use an Octave script to manage the variables.)
Answer: $\frac{dz}{dt} = 10^{-11}((3x^2y + 10xy^2)\frac{dx}{dt} + (x^3 + 10yx^2)\frac{dy}{dt})$, which evaluates to -.0679

114. (Apex 12.5) 23,25
(Apex 12.7) 21,23

115. Consider the graph of $\exp(y/z) = xz^2$ as defining z as an implicit function of x and y. At the point where $x = 0.5$ and $y = 1.2$, find numerical values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Answer: using W.A., $z = 1.93$
$\frac{\partial z}{\partial x} = \frac{-z^2}{ye^{y/z}z^{-2}+2xz} = -1.4723$ and $\frac{\partial z}{\partial y} = \frac{e^{y/z}z^{-1}}{2xz+yz^{-2}e^{y/z}} = .38138$

note: if $dx = .01$, then $dy = 1.4723/.38138(.01) = .0386$ to stay on contour

116. Suppose z is an implicit function of x and y, with $y(z^2 + 2) = xe^z$.

(a) Verify that $(6, 3, 0)$ is on the graph.
Answer: plug in to the equation and you get $6 = 6$

(b) Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ by differentiating implicitly.
Answer: first $\frac{\partial}{\partial x}\left(yz^2 + 2y\right) = \frac{\partial}{\partial x}\left(xe^z\right)$, so $2yz\frac{\partial z}{\partial x} = e^z + xe^z\frac{\partial z}{\partial x}$, and therefore $\frac{\partial z}{\partial x} = \frac{e^z}{2yz-xe^z}$
then $\frac{\partial}{\partial x}\left(yz^2 + 2y\right) = \frac{\partial}{\partial x}\left(xe^z\right)$, so $z^2 + 2yz\frac{\partial z}{\partial y} + 2 = xe^z\frac{\partial z}{\partial x}$, and therefore $\frac{\partial z}{\partial y} = \frac{z^2+2}{xe^z-2yz}$

(c) Find $\nabla z(6, 3, 0)$.
Answer: plug in to the partial derivatives to get $\nabla z = [-1/6, 1/3]$

(d) Find the equation of the tangent plane at that point.
Answer: $z = \frac{-1}{6}(x - 6) + \frac{1}{3}(y - 3)$

(e) Find the equation of the normal line at that point.
Answer: $\ell(t) = \begin{bmatrix}6\\3\\0\end{bmatrix} + \begin{bmatrix}-1/6\\1/3\\-1\end{bmatrix}t$

117. Find the quadratic function $f(x, y) = c_1 + c_2x + c_3y + c_4x^2 + c_5y^2 + c_6xy$ such that ("exactly"):

(a) $f(5, 2) = 10$

(b) $\nabla f(5, 2) = \begin{bmatrix}0\\0\end{bmatrix}$

(c) $\frac{\partial^2 f}{\partial x^2} = 0.70$

(d) The concavity is 0.20 in the direction $\vec{v} = \begin{bmatrix}-10\\3\end{bmatrix}$.

(e) The concavity is 0.20 in the direction $\vec{v} = \begin{bmatrix}-10\\6\end{bmatrix}$.

Hint: start by matching the 2nd derivatives.
Answer: Write $H = \begin{bmatrix}.7 & a\\a & b\end{bmatrix}$ and compute concavity in the two directions using $\frac{\vec{v}\cdot(H\vec{v})}{\vec{v}\cdot\vec{v}}$ to get a system of equations: $70 - 60a + 9b = (109)(.2)$ and $70 - 120a + 36b = (136)(.2)$, giving $H = \begin{bmatrix}.7 & 1.25\\1.25 & 134/45\end{bmatrix}$.
Therefore $c_6 = 1.25$ and $c_5 = 134/90$ and $c_4 = 0.35$.
Then, at the critical point $f_x = c_2 + 2c_4(5) + c_6(2) = f_y = c_3 + 2c_5(2) + c_6(5) = 0$. Solving gives $c_2 = -6$ and $c_3 = -2197/180$. Finally, at $f(5, 2) = 10$ implies $c_1 = 6697/180$. Therefore

$$f(x, y) = 6697/180 - 6x - (2197/180)y + .35x^2 + (134/90)y^2 + 1.25xy$$

118. (Apex 12.8) 5,9,13

119. This function has a local max and a local min. How much higher is the local max than the local min?

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 + 20$$

Answer: the local max is $(-2, 0, 24)$ and the local min is $(0, 2, 16)$, so the difference is 8

120. Suppose the gradient of a function is:

$$\nabla f = \begin{bmatrix} y(y-6)(x+1) \\ x^2y - 3x^2 + 2xy - 6x - 5.5y + 9 \end{bmatrix}$$

Find and classify the critical points (there are 5 of them)
Answer: if $y = 0$, then $-3x^2 - 6x + 9 = -3(x^2 + 2x - 3) = -3(x+3)(x-1) = 0$ gives critical points $(-3, 0)$ and $(1, 0)$.

if $y = 6$, then $6x^2 - 3x^2 + 12x - 6x - 33 + 9 = 3x^2 + 6x - 24 = 3(x^2 + 2x - 8) = 3(x+4)(x-2) = 0$ gives c.pts $(2, 6)$ and $(-4, 6)$.

if $x = -1$, then $y - 3 - 2y + 6 - 5.5y + 9 = -6.5y + 12 = 0$ gives c.pt $(-1, 24/13)$.

The Hessian is $\begin{bmatrix} y(y-6) & 2xy - 6x + 2y - 6 \\ 2xy - 6x + 2y - 6 & x^2 - 2x - 5.5 \end{bmatrix}$.

- $(-3, 0)$, $D < 0$ so saddle
- $(1, 0)$, $D < 0$ so saddle
- $(2, 6)$, $D < 0$ so saddle
- $(-4, 6)$, $D < 0$ so saddle
- $(-1, 24/13)$, $D > 0$, $f_{xx} < 0$, so local max

121. Consider the plane with x, y, and z intercepts of 2, 5, and 3 respectively. There is a point $P(3, 4, 6)$ that is not on the plane.

(a) Find the equation of the plane.
Answer: $15x + 6y + 10z = 30$, or $z = 3 - 1.5x - 0.6y$

(b) Write the objective function $f(x, y)$ giving the squared distance from P to a point (x, y, z) on the plane.
Answer: $f(x, y) = (x-3)^2 + (y-4)^2 + (3 - 1.5x - .6y - 6)^2$

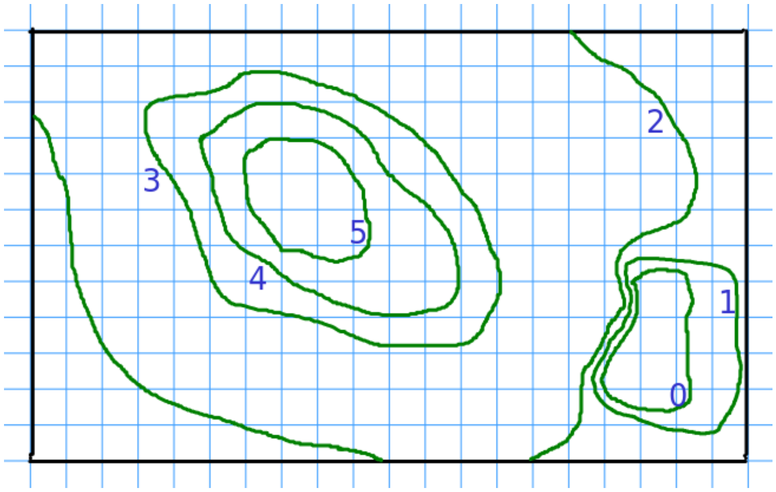
(c) Minimize f by setting $\nabla f = \vec{0}$, and verify that it's a minimum with the 2nd derivative test.
Answer: $\begin{bmatrix} 2(x-3) + -3(3 - 1.5x - .6y - 6) \\ 2(y-4) - 1.2(3 - 1.5x - .6y - 6) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ where $x = -1.11, y = 2.35$

(d) Find the shortest distance from P to the plane.
Answer: plug in to get $z = 3.26$, and the distance is $\sqrt{f(x, y)} \approx 5.21$

122. Find the line $y = a + bx$ that "best" fits the points $\{(0, 2), (5, 0), (3, 3)\}$. The objective function should be the sum of squared y errors as demonstrated in class.
Answer: minimize $f(a, b) = (a-2)^2 + (a+5b)^2 + (a+3b-3)^2$ to get the line $y = 2.58 - 0.342x$

Integration

123. The population density of a 20×12 km city is shown in this contour plot. Densities are in thousands of people per square km. Use a Riemann sum with 4×2 blocks to estimate the city's population. (Notice the lake in the SE corner where density drops to zero.

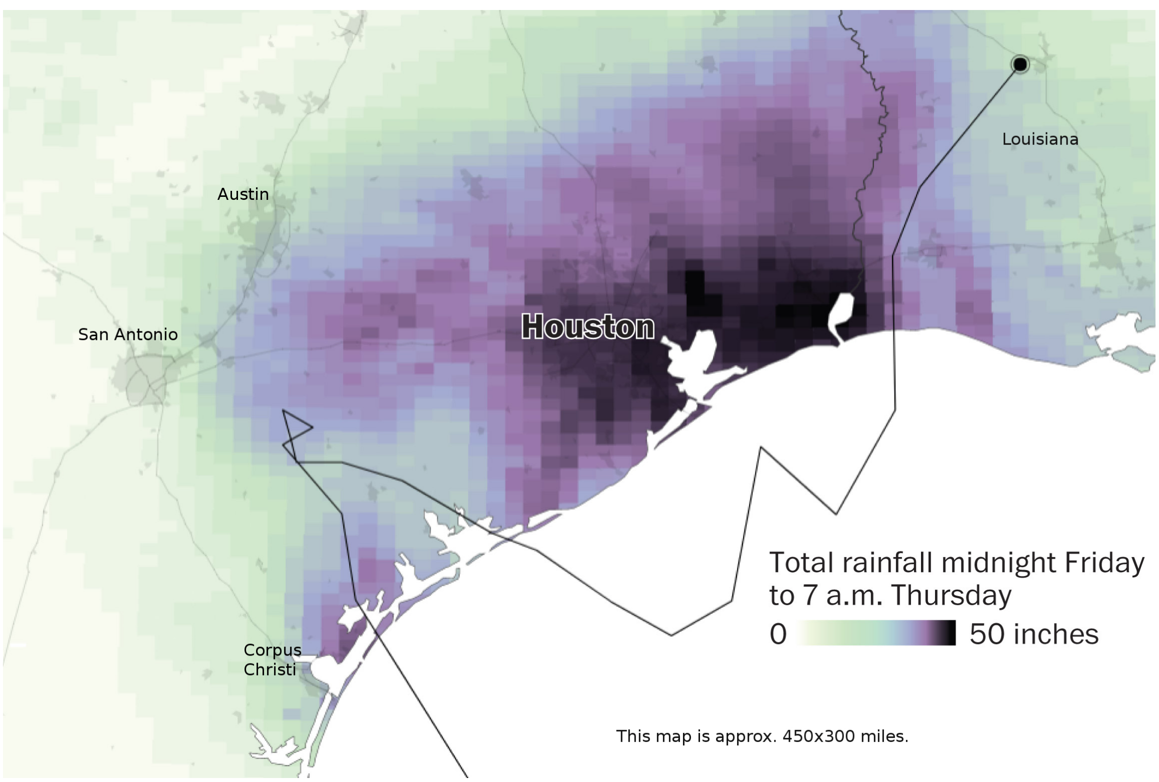


124. How many gallons of rain did Hurricane Harvey dump on the pictured land area?

(a) Divide the 450×300 map into 50×50 squares.

(b) Estimate rainfall at each square's midpoint, and compute a Riemann sum.

(c) Adjust your units so that you can find the total number of gallons of water.
Answer: about 25 trillion



125. (Apex 13.1) 5,13,15,19,21

126. (Apex 13.2) 5,7,11,15,21,25

127. (Apex 13.3) 3,7,13,15

128. Find an EXACT value of a such that $\displaystyle\int_2^5 \left(\int_0^a \frac{x}{y}dx\right)dy = 3$.
Answer: $\int_2^5 \frac{a^2}{2y}dy = \frac{1}{2}a^2(\ln(5) - \ln(2)) = 3$, so $a = \pm\sqrt{6/(\ln(5/2))}$

129. Let $R = [3,8] \times [1,4]$. Use WolframAlpha to evaluate $\displaystyle\iint_R e^{-xy/16}dA$.
Answer: $\int_1^4 \int_3^8 e^{-xy/16}dxdy \approx 6.79$

130. Let $R = [3,8] \times [1,4]$. Consider the integral $\displaystyle\iint_R x\sin(xy)dA$. Write the integral with $dA = dxdy$, and also write it with $dA = dydx$. Evaluate by hand whichever is easier.
Answer: you'd rather do the y integral first, so $\int_3^8 \int_1^4 x\sin(xy)dydx = \int_3^8 \cos(x) - \cos(4x)dx = (\sin(8) - .25\sin(32)) - (\sin(3) - .25\sin(12)) \approx .5762$

131. Suppose a pond has a surface area of 3200 square feet, and an average depth of 3.64 feet. Let $d(x,y)$ be the depth at a given point in the pond's region R. Find $\displaystyle\iint_R d(x,y)dA$.
Answer: $(3200)(3.64) = 11648$ cubic feet

132. Let R be bounded by $|x-12| = 8$ and $|y-9| = 7$. Find the average distance of points in R to the center of that rectangle. (Do the integral in WA)
Answer: $\frac{1}{224}\int_4^{20}\int_2^{16}\sqrt{(x-12)^2+(y-9)^2}dydx \approx 5.747$

133. Let $f(x,y) = \dfrac{5x^3}{(4+x^2y)^{3/2}}$. Do these integrals by hand (with dx on the outside).

    (a) Do an improper integral over the horizonal strip $[0,\infty) \times [0,1]$. Does it converge?
    Answer: $\int_0^\infty \int_0^1 f(x,y)dydx = \int_0^\infty \left(-10x(4+x^2y)^{-1/2}\Big|_{y=0}^{y=1}\right)dx = \int_0^\infty -10x(4+x^2)^{-1/2} + 5x = -10(4+x^2)^{1/2} + \frac{5}{2}x^2\Big|_{x=0}^{x=\infty} = \infty$ (it diverges)

    (b) Do an improper integral over the vertical strip $[0,1] \times [0,\infty)$. Does it converge?
    Answer: $\int_0^1 \int_0^\infty f(x,y)dydx = \int_0^1 \left(-10x(4+x^2y)^{-1/2}\Big|_{y=0}^{y=\infty}\right)dx = \int_0^1 5xdx = 5/2$

    (c) Integrate over the square $[0,a] \times [0,a]$. Your answer should be a function of a.
    Answer: $\int_0^a \int_0^a f(x,y)dydx = \int_0^a \left(-10x(4+x^2y)^{-1/2}\Big|_{y=0}^{y=a}\right)dx = \int_0^a -10x(4+ax^2)^{-1/2} + 5xdx = \frac{-10}{a}(4+ax^2)^{1/2}\Big|_{x=0}^{x=a} + 5a^2/2 = \frac{-10\sqrt{4+a^3}}{a} + \frac{20}{a} + 5a^2/2$

    (d) Find the value of a that makes that last integral evaluate to 1.
    Answer: (WA) 1.55784

134. Let $f(x,y) = xe^{(y-x^2)/5}$, and let $R = [0,a] \times [0,b]$.

    (a) Show that this function is **separable**, i.e. $f(x,y) = g(x)h(y)$.
    Answer: $g(x) = xe^{-x^2/5}$ and $h(y) = e^{y/5}$

    (b) Write $\displaystyle\iint_R f(x,y)dA$ as a function $V(a,b)$.

    (c) Find $\nabla V(a,b)$ where $a = 1$ and $b = 3$.

135. Evaluate $\int_0^2 \int_{y^2}^{2y} (4x - y)dxdy$ by hand

Answer: $\int_0^2 (6y^2 - 2y^4 + y^3)dy = \frac{36}{5}$

136. Sketch the region of integration, reverse the order of integration, and evaluate by hand

$$\int_0^1 \int_{2x}^2 e^{y^2} dydx$$

Answer: $\int_0^2 \int_0^{\frac{y}{2}} e^{y^2} dxdy = \int_0^2 \frac{y}{2} e^{y^2} dy = \frac{1}{4} \int_0^4 e^u du = \frac{1}{4}(e^4 - 1)$

137. Evaluate $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} dxdy$ by hand.

Answer: $\int_0^1 3y^2 e^{y^3} - 3y^2 dy = e - 2$

138. Integrate $f(x, y) = e^x \ln y$ over the region in the 1st quadrant, above $x = \ln y$, having $|y - 3| < 2$. Do the inner integral by hand, but use W.A. for the outer integral.

Answer: $\int_1^5 \int_0^{\ln(y)} e^x \ln(y)dxdy = \int_1^5 \ln(y)(y - 1)dy \approx 10.07$

139. Consider a circle of diameter D, described by $r = D\sin(\theta)$. Let P be a point on the circle's boundary. Find the average distance from a point inside the circle to the point P.

Answer: $\dfrac{\int_0^\pi \int_0^{D\sin(\theta)} rrdrd\theta}{\pi D^2/4} = \dfrac{4D^3/9}{\pi D^2/4} = \dfrac{16D}{9\pi}$