1. Use markdown in Jupyter to format this quote nicely, and include a picture of Marcus Aurelius.

   In the series of things, those which follow are always aptly fitted to those which have gone before: for this series is not like a mere enumeration of disjointed things, which has only a necessary sequence, but it is a rational connection: and as all existing things are arranged together harmoniously, so the things which come into existence exhibit no mere succession, but a certain wonderful relationship.
   – Marcus Aurelius

2. Let $a_n = n(n+1)$.
   (a) Write out the terms of the sequence $\{a_n\}_{n=0}^{5}$.
   (b) Write out the sequence of partial sums $s_n = \sum_{k=0}^{5} a_k$.
   (c) Find a formula (in terms of n) for $s_n = \sum_{k=0}^{n} a_k$. (Hint: google "sum of squared integers".)
   Write your answer as a factored polynomial in n.
   (d) Solve $a_n \geq 1000$ for n.
   (e) Solve $s_n \geq 1000$ for n.

3. Let $\{a_k\}_{k=1}^{\infty}$ be an infinite sequence, having partial sums:

$$s_n = \frac{100}{3 + 16^{1/n}}$$

   (a) Fill out this table (on your own paper):

n	a_n	s_n
1		
2		
3		
4		
5		

   (b) Use a one line calculation to find a_{30} directly from the s_n formula.
   (c) Compute the infinite series $\sum_{k=1}^{\infty} a_k$ by evaluating a limit.
   (d) What must be the value of $\lim_{k\to\infty} a_k$? Explain with a complete sentence.

4. Derive the formula for the sum of the infinite geometric series $\sum_{n=0}^{\infty} r^n$, with $|r| < 1$.
   (a) Let $s_N = 1 + r + r^2 + \cdots + r^N$ be a partial sum. Compute $(1-r)s_N$.
   (b) Solve for s_N.
   (c) Evaluate $\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} s_N$

5. Consider a rational number with repeating decimal $x = 2.162\overline{73}$.
   (a) Write it as $x = a + b(1 + r + r^2 + \cdots)$.
   (b) Plug in the formula for the geometric series.
   (c) Write using fractions, and get a common denominator to express x as the ratio of two integers.

6. Suppose you manage a fleet of trucks that get highway fuel economy modeled by:

$$y = \frac{x}{2 + (.02x)^4}$$

where x is speed in MPH, and y is your gas mileage in MPG.

(a) Plot y as a function of x, and use calculus to find the speed that maximizes gas mileage.

(b) If gas is \$2.00 per gallon, and drivers are paid \$15 per hour, how fast should they drive to minimize total monetary costs? What would be the fuel economy? What is the total cost per mile? Give your answers in the form of a paragraph.

(c) Suppose gas is G dollars per gallon, and drivers are paid T dollars per hour. Let X_o be the optimal speed to minimize total costs. Write X_o as a function of G and T.

(d) Plot the contours $X_o(G,T) = \{55, 65, 75\}$.

(e) If gas is \$ 2 per gallon, what value does a 65 MPH speed limit put on the drivers' time?

(f) Find $\frac{\partial X_o}{\partial G}(2, 15)$ and $\frac{\partial X_o}{\partial T}(2, 15)$.

(g) Write the linear approximation (tangent plane) to X_o at $G = 2$ and $T = 15$.

(h) Use only $\nabla X_o(2, 15)$ to answer this problem. If gas goes up 10 cents, then approximately what increase in driver pay would leave the optimal speed unchanged?

(i) Suppose gas goes up 25 cents and the drivers get a \$5 raise. (So now $G = 2.25$ and $T = 20$).

  i. You should instruct the drivers to go _______ MPH (faster | slower).

  ii. Use the linear approximation to answer the previous question, and contrast the approximate and exact answers.

  iii. Using the exact optimal speed, find the new cost per mile.

  iv. If your your drivers log about 300 thousand miles per year, then the increase in gas and labor prices will increase your total expenses by _______ thousand dollars.

7. Suppose you stack an infinite number of spherical snowballs. The bottom snowball has a radius of 2 feet, and each subsequent snowball has a 30% smaller radius than the one below it. Use the geometric series formual to:

   (a) Find the total height of your stack.

   (b) Find the total volume.

8. Investigate the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)}$.

   (a) Use Sage to produce a table of partial sums up to $n = 100$.

   (b) Use a program to compute the partial sum up through $n = 10^3, 10^4, 10^5, 10^6$.

   (c) Discuss the behavior of the partial sums. Do you think the series converges?

9. Find a polynomial of smallest possible degree that satisfies:
   $f(0) = 7$, $f'(0) = 4$, $f''(0) = -3$, and $f'''(0) = 5$.

10. Find a polynomial of smallest possible degree that satisfies:
    $f(2) = 7$, $f'(2) = 4$, $f''(2) = -3$, and $f'''(2) = 5$.

11. An amusement park ride is bobbing up and down vertically on a tower. Let $y(t)$ describe the position (feet) at time t seconds. Suppose that over the interval $t \in [30, 32]$, y is a 6th degree polynomial, and when $t = 30$ you have:

    - position: $y(30) = 40$
    - velocity: $y'(30) = 22$
    - acceleration: $y''(30) = 8$
    - jerk: $y'''(30) = -35$
    - snap: $y^{(4)}(30) = 15$
    - crackle: $y^{(5)}(30) = 0$
    - pop: $y^{(6)}(30) = -75$

    (a) Find the formula for $y(t)$.

    (b) Plot the graph for $t \in [30, 32]$.

    (c) Find the position when $t = 32$.

    (d) Find the maximum height reached over the time interval.

12. Here is some terminology about optimization.

    - The function you want to maximize or minimize is called the **objective function**.
    - An **extremum** can be either a maximum or a minimum, and is locally the biggest or smallest output to the function.
    - An extremum is **global** if it is the biggest or smallest output on the entire domain of the objective function.
    - The **argmax** of a function are the arguments (inputs) to the function at which a maximum occurs.
    - On a given domain, the notation "$\max f$" refers to the global maximum output, while "$\arg\max f$" refers the input(s) at which that maximum occurs.

    Consider the objective function $f(x) = 2x^3 - 21x^2 + 60x$ on the domain $[1, 7]$.

    (a) List the (x, y) coordinates of all local extrema.
    (b) $\max f$
    (c) $\arg\max f$
    (d) $\min f$
    (e) $\arg\min f$

13. Show all steps to find and classify ("by hand") the critical points of:

$$f(x, y) = xy^2 - 2y^2 - 2x^3 - 9x^2 + 24x$$

14. Let $z = f(x, y)$ be an objective function such that

$$\nabla f(x, y) = \begin{bmatrix} 2xy - 10x - 4y + 20 \\ x^2 - 4x + 3(y - 4)^2 \end{bmatrix}$$

    Without using a computer to do the work:

    (a) Find the Hessian matrix $H(x, y)$.
    (b) Show how to find the critical points. First factor f_x and set it to zero.
    (c) Apply the 2nd derivative test to each critical point.
    (d) Anti-differentiate to find a function f that has the given gradient.
    (e) Find the exact difference in z-values between the the local max and the local min.

15. Suppose that on $\mathbb{R}^2$, temperature is given by

$$T(x, y) = \exp\left(\frac{12xy - x^2 - y^4}{50}\right)$$

    degrees kelvin. Follow these steps to find the extreme temperature(s).

    (a) Use software to plot the surface of T over the interval $[-100, 100] \times [-10, 10]$.
    (b) Let $z = 50\ln(T)$. Explain why $\arg\max z = \arg\max T$.
    (c) Find the critical point(s) of z, and classify them using the 2nd derivative test.
    (d) What is the maximum temperature, and where does it occur ?
    (e) What is the minimum temperature, and where does it occur ? Explain.
    (f) What is the temperature at the saddle point ?

16. Consider this function:

$$f(x,y) = (x^2 - 1)^2 + (x^2 - e^y)^2$$

(a) Plot the surface on $[-2,2] \times [-2,2]$.

(b) Find the critical points.

(c) Apply the 2nd derivative test to each critical point to classify it.

(d) What is strange about your finding? Is such a result possible for a continuous function of only one independent variable?

17. Find constants $a < b$ such that $\int_a^b (6 - x - x^2)dx$ is largest.

(a) Visualize the answer using Calc I only. Draw a picture to illustrate.

(b) Write your objective function as $z(a,b)$, then find and classify the critical points.

18. A rectangular shaped building should have total volume of 12,000 cubic feet. Annual heating and cooling costs depend on the surface area as follows:

- \$3 per square foot for the roof
- \$1 per square foot for the floor
- \$5 per square foot for the back and front
- \$4 per square foot for the left and right sides

Let ℓ, w, h represent the length, width, and height of the building. Your goal is to find dimensions that minimize the annual heating and cooling costs.

(a) Write the cost as a function $C(\ell, w, h)$.

(b) Apply the volume constraint by substituting for h, so that you have $C(\ell, w)$.

(c) Find $\nabla C(\ell, w)$, and set it to zero to find ℓ and w.

(d) Check that the critical point minimizes C by using the 2nd derivative test.

(e) State the optimal dimensions, and the resulting cost.

19. Consider the parabola $f(x) = \frac{1}{2}x^2$. Let $P(2,5)$ be a point (inside the bowl). Let Q be the point on the graph of f that is closest to P.

    (a) Write an objective function z as the squared distance from P to a point on the graph of f.

    (b) Find all critical points, and classify them using the 2nd derivative test.

    (c) Find the point Q, and the distance from P to Q.

    (d) Verify that $\vec{PQ}$ is parallel to the vector $\vec{n} = [\frac{df}{dx}, -1]$ normal to the graph of f at Q.

    (e) Plot f and the point P. Also plot z. Identify each critical point of z with the corresponding point on f, and illustrate orthogonality of f with the vector from that point to P. (you can print out the function graphs and elaborate by hand.)

20. Consider the the paraboloid $f(x,y) = \frac{1}{2}x^2 + y^2$. Let $P(2,1,5)$ be a point (inside the bowl). Let Q be the point on the graph of f that is closest to P.

    (a) Write an objective function z as the squared distance from P to the graph of f.

    (b) Find all critical points, and classify them using the 2nd derivative test.

    (c) Find the point Q, and the distance from P to Q.

    (d) Verify that $\vec{PQ}$ is parallel to the vector $\vec{n} = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1]$ normal to the graph of f at Q.

21. Consider the paraboloid $z = 150 + 7x + 10y + 2xy - x^2 - 3y^2$.

    (a) Use software to generate a contour plot over $[0,10] \times [0,10]$.
    Make it fairly large, and include the contours $z = 0, 50, 100, 125, 140, 155, 170, 180, 190, 195$.
    Put an 'X' at the apparent location of the surface's peak.

    (b) Find $\nabla z(x,y)$.

    (c) Solve the 2×2 system of equations $\nabla z = \vec{0}$.

    (d) What is the value of z at the surface's peak ?

    (e) A bird flys above the surface along a path with $y = 2x$. Sketch the bird's path on your contour plot, and place an 'B' on the plot at the point under the bird's path where the elevation of z appears to be greatest.

    (f) Sketch ∇z at point 'B', and describe its direction in relation to the contour plot.

    (g) Substitute and then use basic calculus to find the exact point (x, y, z coordinates) under the bird's path that maximizes z.

22. The second derivative test is inconclusive if the discriminant $D = 0$. Consider these two examples, which have a single critical point at $(0,0)$.

    (a) $z(x,y) = x^2 + 4xy + 4y^2 + x^4$

        i. Find the Hessian matrix at $(0,0)$, and show that $D = 0$.

        ii. Create a contour plot on $[-1,1] \times [-1,1]$. Include contours $z = 0.01, 0.1, 0.2, 0.5, 1, 2, 3, 5, 8$.
        Does the critical point appear to be a max, min, or saddle ?

        iii. Factor the quadratic part of $z(x,y)$ to show that $z(x,y) \geq z(0,0)$.

        iv. Is $(0,0)$ a max, min, or saddle? Explain how you now know for sure.

    (b) $z(x,y) = x^2 + 4xy + 4y^2 - x^4$

        i. Find the Hessian matrix at $(0,0)$, and show that $D = 0$.

        ii. Create a contour plot on $[-1,1] \times [-1,1]$. Include contours $z = -1, -0.5, 0, 0.1, 0.1, 0.5, 1, 2, 5$.
        Does the critical point appear to be a max, min, or saddle ?

iii. Show that if $x = 0$ near the origin, then $z \geq 0$.
And if $x = -2y$ near the origin, then $z \leq 0$.

iv. Is $(0, 0)$ a max, min, or saddle? Explain how you know for sure.

23. Big Bang Theory: Season 8, Episode 2:

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Sheldon: Do you know how to integrate X squared times E to the minus X,
         without looking it up?

 Howard: I'd use Feynman's trick; differentiate under the integral sign.
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(a) For $t > 0$, define $f(t) = \int_0^\infty e^{-tx} dx$. Evaluate this integral to have a simple formula for $f(t)$.

(b) Use the "Leibniz Rule" to differentiate both sides.

$$\frac{d}{dt} f(t) = \int_0^\infty \frac{\partial}{\partial t}(e^{-tx}) dx$$

(c) Plug in $t = 1$ to get the value of $\int_0^\infty xe^{-x} dx$.

(d) Apply the Leibniz Rule again.

$$\frac{d^2}{dt^2} f(t) = \int_0^\infty \frac{\partial^2}{\partial t^2}(e^{-tx}) dx$$

(e) Plug in $t = 1$ to get the value of $\int_0^\infty x^2 e^{-x} dx$.

(f) Apply the Leibniz Rule again.

$$\frac{d^3}{dt^3} f(t) = \int_0^\infty \frac{\partial^3}{\partial t^3}(e^{-tx}) dx$$

(g) Plug in $t = 1$ to get the value of $\int_0^\infty x^3 e^{-x} dx$.

(h) Following this pattern, find a formula for $F(n) = \int_0^\infty x^n e^{-x} dx$.

(i) You know that $3! = 6$ and $4! = 24$, but what is 3.5 factorial ? Interpolate the factorial function by defining $n! = F(n)$ for non-integer values of n. Evaluate $(3.5)!$.

(j) Use trial-and-error to solve $F(n) = 10$.

(k) Verify that $0! = F(0) = 1$.

24. Optimize $z = x^3 + 8y$ subject to the constraint $\frac{x^2}{4} + \frac{y^2}{2} = 1$.

   (a) Write the Lagrange system of equations.

   (b) Plot the constraint along with the contour plot of the objective function. Mark by hand the approximate locations of the Lagrange solutions.

   (c) List each of the Lagrange solutions, and label which ones maximize or minimize z.

25. Optimize $f(x, y) = e^{x^2 y}$ subject to the constraint $x^4 + y^4 = 16$.

   (a) Plot the constraint.

   (b) By inspecting the objective function, in which quadrants would you expect to find the maximums ? What about the minimums ?

   (c) Write the Lagrange system of equations, and use a computer to solve them.

   (d) Report the max and min values of f, and where they occur.

26. Purchase b units of biscuits, and c units of coffee. Your breakfast satisfaction is given by

$$f(b, c) = \frac{100(b/10 + c/6)^1}{(1 + (b/10 + c/6)^3)(1 + (b/5 - c/3)^2)}$$

   Use Sage for this problem, and include the contour and constraint plots.

   (a) With an unlimited budget, find b and c to maximize f.

   (b) If one unit of biscuit is \$ 1, and one unit of coffee is \$ 2, optimize f subject to the constraint that you will spend your budget of \$ 7. Report how many units of biscuit and coffee you will purchase, and the satisfaction it will give you.

   (c) What if the prices of biscuits and coffee flipped (biscuits are now \$ 2, and coffee is \$ 1) ? Update your optimization and report the new values of b, c, and f.

   (d) Now suppose the price of biscuits goes up 20¢ to \$1.20. Experiment to find a price decrease in coffee (from \$2) that would leave you just as satisfied as before.

27. Consider this function defined on $\mathbb{R}^2$.

$$f(x, y) = e^{3x} + y^3 - 3ye^x$$

   (a) Find the gradient.

   (b) Show that there is only one critical point, and apply the 2nd derivative test to show that it is a local min.

   (c) Is it the global minimum? Explain.

   (d) Minimize f subject to the constraint that $x^2 + y^2 = 4$. Plot the constraint and contour plot on $[-3, 2] \times [-3, 2]$. Starting with $[-2, -1, -.75, -.5, 0, 1, 4, 20, 50]$, add one more contour that corresponds (approximately) to the minimum value of f on the const raint.

   (e) Label on your graph the local minimizer in part (b), and the constrained minimizer in part (d).

28. Let $f(a, b) = \int_a^{a+1} \int_b^{b+1} (4x + y + xy - x^2 - 3y^2) dy dx$. This represents the volume under a paraboloid, above a unit square of the xy-plane.

   (a) Produce a contour plot of the integrand.

   (b) At what values of x and y is the integrand maximized?

   (c) Find $\arg\max f$ and $\max f$.

   (d) Draw the 1×1 square of integration on your contour plot. Explain how it relates to the peak of the paraboloid.

29. Recall the Carson-Newman seal pictured here:

with boundary described in polar coordinates by $C : r(t) = 65 - 3\cos(8t)$ for $t \in [0, 2\pi]$. A room with this shape (having flat floor and vertical walls) is constructed with the height of the ceiling given by

$$z(x, y) = 20 - \frac{1}{4000}((x - 2y)^2 + (x - 30)^2 - y^2 + .1x(x + 20)(x - 50))$$

Use the (cell demo) Sage script to answer all of these questions.

(a) Produce a plot of the room along with contours of the ceiling.

(b) Find and classify critical points of z inside the room.

(c) Find the maximum and minimum heights along the wall (and their locations).

  i. Mark the Lagrange criteria solutions on your plot.

  ii. Plot z as a function of t.

  iii. Use Sage's `find_root` command to hone in on the min and max (where $\frac{dz}{dt} = 0$), and find the corresponding values of t, x, y, z.

(d) Find the perimeter of the floor.

(e) Find the perimeter of the ceiling.

(f) Find the surface area of the floor.

(g) Find the surface area of the ceiling.

(h) Find the surface area of the walls.

(i) Find the volume of the room.

(j) Which is greater, the average height of the wall, or the average height of the ceiling ?

(k) Evaluate $\int_C \begin{bmatrix} -.5y \\ .5x \end{bmatrix} \cdot d\vec{r} = \frac{1}{2} \int_C xdy - ydx$. What does this give you?

30. Consider a room with base bounded by the $y = x$ (the "Gordon line") and $y = c - x^2$. The room has vertical walls, and a flat floor covering 1000 square feet. The ceiling above has height given by

$$z(x, y) = 10 + 20\exp(-3x/25 + x^3/625 + y^2/4000 - (y/60)^4)$$

You may modify the starter Sage cell demo script to answer these questions.

(a) Find the value of c to the nearest integer.

(b) Produce a plot of the room along with contours of the ceiling.

(c) Find the maximum and minimum heights of the ceiling, and where they occur. You must check the interior (2nd derivative test) and the boundary (set $\frac{dz}{dt} = 0$ to find the Lagrange point(s)).

(d) Find the perimeter of the floor.

(e) Find the perimeter of the ceiling.

(f) Find the surface area of the walls.

(g) Find the surface area of the ceiling.

(h) Find the volume of the room.

(i) Compare the average height of the walls with the average height of the ceiling.

(j) Dr. Emmons (include photo) has outfitted you with a suit so that when you're in the room, the force field:

$$\vec{F}(x, y, z) = \begin{bmatrix} 2x \\ -y \\ \gamma xy \end{bmatrix}$$

Newtons/kg acts on you.

i. Find the constant γ so that you would feel weightless at the southeast corner, i.e. the z-component of $\vec{F}$ counters the $-9.81\vec{k}$ /kg gravitational force.

ii. Add vector field (just the x,y components of $\vec{F}$) and streamline plots to your previous graph of the room.

iii. Explain why $\vec{F}$ acts conservatively if you move in a horizontal plane so that $\frac{dz}{dt} = 0$.

iv. Compute the work $\int_C \vec{F} \cdot d\vec{r}$ where C is a piecewise linear path (along the floor) starting from the northernmost point in the room, going to the southeast corner, then to the spot directly under the highest point of the ceiling, and then going due north until you reach the wall. (sketch that path on your graph, and explain why the work integral should be positive).

v. Compute the work $\int_C \vec{F} \cdot d\vec{r}$ where C is a path traversing the perimeter of the floor.

vi. Compute the work $\int_C \vec{F} \cdot d\vec{r}$ where C is a path traversing the perimeter of the ceiling. (say you go around counter-clockwise, as observed looking up at the ceiling)

31. A fortified city is built on a flat plain, its walled boundary described by the positively oriented simple closed curve C:

$$\begin{aligned} x(t) &= -\sin(t)(2 + 40e^{-2t^2}) \\ y(t) &= 20\cos(t/2) \end{aligned}$$

for $t \in [-\pi, \pi]$, and x and y measured in kilometers. The wall's height (in meters) is given by:

$$h(x, y) = (25 - x)e^{-y/20}$$

(a) Do a 2D parametric plot of C, with arrows to show the orientation.

(b) Include a picture of the city's king.

(c) Find the maximum width of the city (from west to east). Do this using calculus, not by just approximating from the graph.

(d) Find the perimeter of the city at the base of the wall.

$$P = \oint_C 1 ds$$

"On the seventh day, march around the city seven times, with the priests blowing the trumpets." –Joshua 6:4

(e) A sentry follows the protesters around the city, but walking along the top of the wall. Plot the height of the wall as a function of t. Find all local extrema for the height of the wall (using calculus). Report the (x, y, z) locations of those extrema.

(f) To keep out the sound of the trumpets, the king has ordered a sound-proof layer to cover the wall's entire exterior. Find the surface area of the wall in square meters.

$$W = 1000 \oint_C h(x, y) ds$$

(g) Find the area of the city in square kilometers.

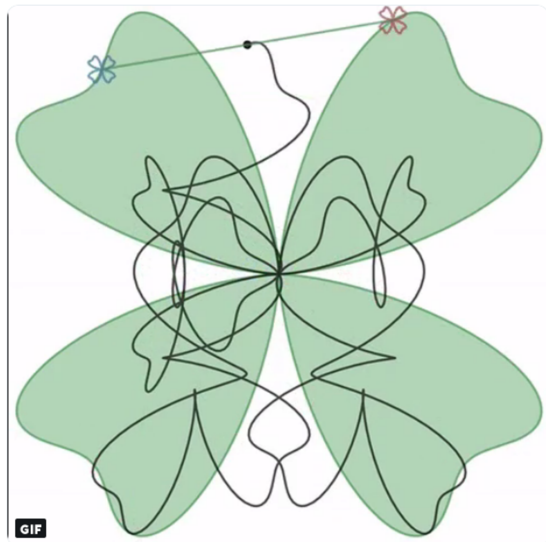
$$A = \oint_C x dy = \oint_C -y dx$$

Show that both calculations yield the same result.

(h) Real estate value in the city is given by $V(x, y) = a + b(5xy + 90x - x^2 y)$, for some positive constants a and b. Find the global extrema: $\arg\max V$ and $\arg\min V$.

32. Inspired by the tweet `https://twitter.com/LukeSelfwalker/status/842722596072361986`.

The path of the midpoint of 2 clovers that travel in opposite directions around a clover. First shows 5 times faster, then 100 times faster.



Use the Sage Cell demo to help you do the computations and graphs.

(a) Design parametric equations that describe a walking path (it may intersect itself). Set them up so that $t = 0$ corresponds to the initial point. The equations must be periodic (and differentiable) so that after one lap, the same path is traced again and again.

(b) Suppose Alex starts walking in one direction, and Beth starts walking in the opposite direction at the same speed. For simplicity, let $\vec{r}_b(t) = \vec{r}_a(-t)$, without maintaining constant speed. Their dog, Clif, moves so that he stays exacly half-way between Alex and Beth.

  i. Find parametric equations for the dog's path.

  ii. Plot the dog's path overlayed onto the walking path.

  iii. Find the distance traveled by the dog if Alex makes one complete lap.

(c) Now suppose Alex and Beth start off in the same direction, but Beth is jogging at double the speed of Alex. (i.e. half the period) Do the same three steps.

(d) Generally, suppose Beth's initial velocity is b times Alex's, and Clif stays w percent of the way from Alex to Beth. (so you've already considered $w = 50\% = .5$ with $b = -1$ or $b = 2$). Let $D(b, w)$ be the distance traveled by the dog for the given values of b and w.

  i. Experiment and produce a graph of the paths for your favorite values of b and w. Create a full-page color picture and make two copies - one for the front of your portfolio binder, and the other to display as a gallery in DSC 110 (or the Creek Cafe?)

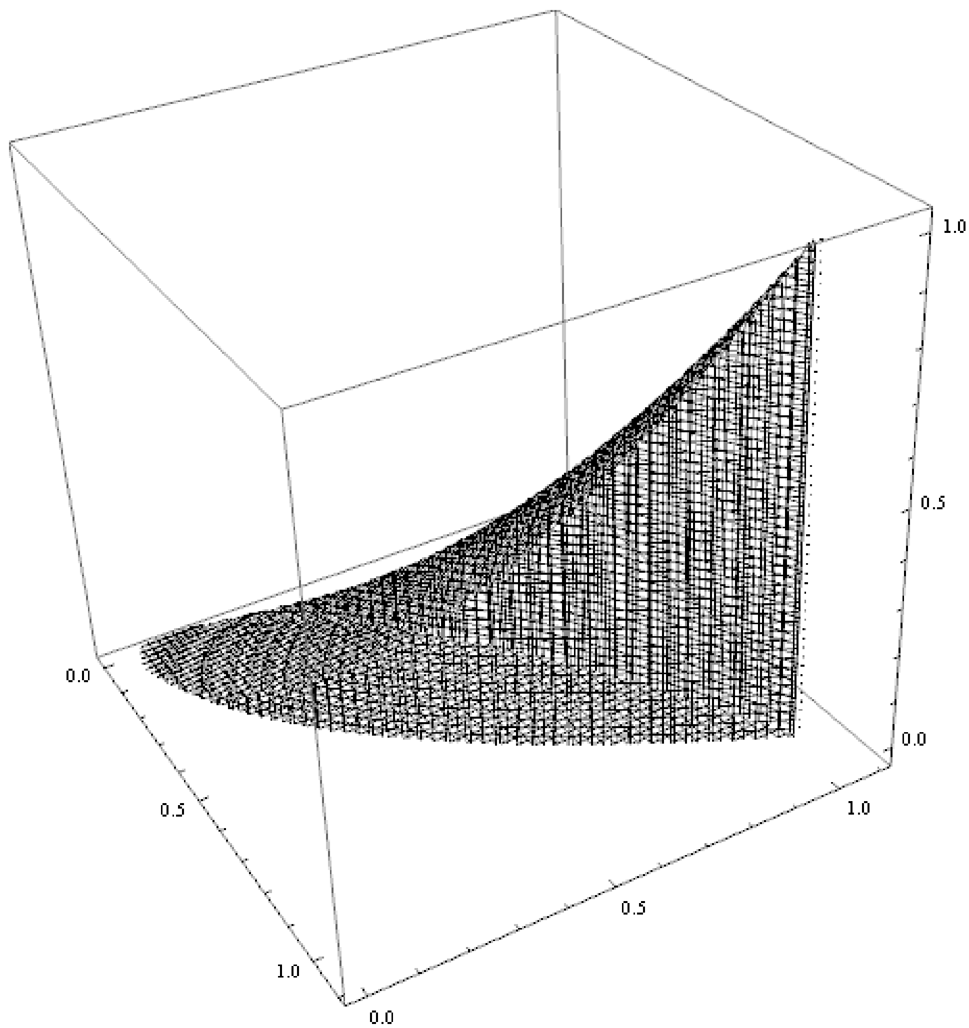
  ii. For $w = .5$, and using a table of points for $b = -5, -4.9, ..5$, plot a graph of $D(b, 0.5)$ with respect to b. Make note of its features.

  iii. Use difference quotients to estimate $\nabla D(-3, .25)$.

33. Recall the Emmons room.

   (a) Find the centroid $(\overline{x}, \overline{y}, \overline{z})$.
   (b) Suppose the room is filled with a material having density $\delta(x, y, z) = (x + 10)e^{-z/50}$. Find the mass, and the center of mass.
   (c) Find the average distance from a point $(x, y, 0)$ in the room to the origin.
   (d) Find the average distance from a point (x, y, z) in the room to the origin.
   (e) Find the average (weighted by δ) distance from a point (x, y, z) in the room to the origin.

34. A solid S (pictured here) is bounded by $y = x^2$, $x = y^2$, $z = 0$ and $z = xy$.



   (a) Write the volume of S as a triple integral, and evaluate with software.
   (b) Suppose that density is proportional to the distance from the origin. If the mass of S is 10, then what is the density at $(.7, .8, .4)$?
   (c) Now suppose that density is inversely proportional to the distance to the origin. Find the center of mass.

35. Let R be the unit disc $x^2 + y^2 \leq 1$. Let S be the square $[-1, 1] \times [-1, 1]$. For each part, compute answers for each of these situations:

- points are chosen "uniformly" (i.e. constant density).
- points are chosen according to the density $\delta(x, y) = x^2 + y^2$. This means that points nearer the origin are less likely to be selected.
- points are chosen uniformly, but only from the 1st quadrant.

Set up the integrals in the most clear and succinct manner. Explain your thought process with words and illustrations (e.g. Desmos)

(a) Find the center of mass (i.e. the average values of x and y) of R.

(b) Find the average distance from a point on the boundary of R to the corner $(1, 1)$ of S.

(c) Find the average distance between two points on the boundary of R.

(d) Find the average distance from a point in R to the origin.

(e) Find the average distance from a point on the boundary of R to a point in R.

(f) Find the average squared distance from a point on the boundary of R to a point in R.

(g) Find the average distance between two points in R.

(h) Find the average distance between a point on the boundary of R and a point on the Gordon line inside R.

(i) Find the average distance between a point in R and a point on the Gordon line inside R.

(j) Find the average distance between a point in R and a point inside S.

(k) Find the average distance between a point in R and a point on the boundary of S.

36. An object undergoes accleration $\vec{a}(t) = \begin{bmatrix} e^{-t/3} \\ -2 \end{bmatrix}$. Given initial position $(0, 50)$ at $t = 0$, and terminal position $(30, 0)$ at time $t = 12$:

   (a) Find and plot the position function $\vec{r}(t)$ for $t \in [0, 12]$.

   (b) Find the position at time $t = 3$.

   (c) At what time is the y-coordinate maximized ?

   (d) How far does the object travel for $t \in [0, 12]$?

   (e) At time has the object traveled half of that distance ?

37. Consider a race course having path:

$$x = \sin(t/25), y = \cos(t/50)$$

   with x, y in kilometers.

   (a) Plot the path (Geogebra setup) for $t \in [0, T]$. For what value of T does it finish one lap?

   (b) Write and evaluate the integral to compute the distance traveled in one lap.

   (c) At what value(s) of t would the tangent line take you to the point $(2, 2)$?

      i. Sketch those line(s) on your graph.

      ii. Write an equation and solve it for t.

      Hint: set $\vec{r}(t) + c\,\vec{v}(t) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, and eliminate the c.

   (d) Mark the first point where the graph has maximum curvature. Using Geogebra find:

      i. the parameter t

      ii. the (x, y) coordinates

      iii. the curvature κ

      iv. the radius of curvature R

      v. the equation of the osculating circle

   (e) If a race car can hold the road as long as $a_N \leq 9\ m/s^2$, then what is the fastest it can go around the sharpest turn ? Give your answer in kilometers per hour.

38. Consider the graph of $y = x^a$ for $a \geq 0$.

   (a) Find the curvature, $\kappa(a, x)$, of the graph in terms of a and x.

   (b) Use a Desmos slider to make a plot of y and κ for each of these values of a: $0, 0.2, 0.8, 1, 1.2, 1.8, 2, 2.2, 3, 4, 8, 16$

   (c) For a given a, set $\dfrac{d\kappa}{dx} = 0$ to find the location of maximum curvature. Do the calculus/algebra yourself. Call the x-value where max curvature occurs $\mu(a)$.

   (d) Find $\lim_{a\to\infty} \mu(a)$ and $\lim_{a\to\infty} \kappa(a, \mu(a))$.

   (e) Find $\lim_{a\to 1-} \mu(a)$ and $\lim_{a\to 1-} \kappa(a, \mu(a))$.

   (f) Find $\lim_{a\to 1+} \mu(a)$ and $\lim_{a\to 1+} \kappa(a, \mu(a))$.