Basics - Spr 2019

1. an (infinite) sequence is a list, and the corresponding sequence of partial sums keeps a running total
2. geometric sequence $\{a_n\} = \{\frac{1}{2^n}\} = \{1, \frac{1}{2}, \frac{1}{4}, \cdots\}$

index	a_n	s_n
0	1	1
1	1/2	3/2
2	1/4	7/4
$\vdots$	$\vdots$	$\vdots$
∞	0	2

   In the limit, the sequence converges to 0, and the series converges to 2. We write $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

3. If a sequence/series doesn't converge, we say it diverges. If a series converges, the sequence must converge to zero.
4. the geometric series converges to $\frac{1}{1-r}$ if $|r| < 1$; otherwise it diverges.
5. the harmonic series $\sum \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ diverges.
6. the p-series $\sum \frac{1}{n^p}$ converges if $p > 1$; otherwise it diverges (by comparing to corresponding integral)
7. Quiz:
   (a) does $\sum \frac{1}{\sqrt{n}}$ converge ?
   (b) does $\sum(\sqrt{2})^{-n}$ converge ?
   (c) sum the infinite geo. series $4032 - 3192 + 2527 - \cdots$
   (d) if $s_n = \frac{5n}{n+5}$, find a_9
8. $f(x, y) = 3x^2y + y^3 - 3x^2 - 2y^2 + 20$
   (a) $\nabla f = \begin{bmatrix} 6xy - 6x \\ 3x^2 + 3y^2 - 4y \end{bmatrix}$ and $H = \begin{bmatrix} 6y - 6 & 6x \\ 6x & 6y - 4 \end{bmatrix}$
   (b) to find critical points, set $\nabla f = \vec{0}$
   $6x(y - 1) = 0$ implies $x = 0$ or $y = 1$
   if $x = 0$, then $3y^2 - 4y = y(3y - 4) = 0$, so $y = 0$ or $y = 4/3$
   if $y = 1$, then $3x^2 - 1 = 0$, so $x = \pm\sqrt{1/3}$

   (c) apply the 2nd derivative test at each critical point:
   - $(0, 0)$: $D = (-6)(-4) - (0)^2 > 0$, and since $f_{xx} < 0$, we are concave down, a local max
   - $(0, 4/3)$: $D = (2)(4) - (0)^2 > 0$, and since $f_{xx} > 0$, we are concave up, a local min
   - $(\sqrt{1/3}, 1)$: $D = (0)(2) - (6\sqrt{1/3})^2 < 0$, so this is a saddle (CU or CD depends on the direction)
   - $(-\sqrt{1/3}, 1)$: $D = (0)(2) - (6\sqrt{1/3})^2 < 0$, so this is also a saddle
9. contours loop around a local min or max, and make a cross at a saddle point;
   the gradient vectors point uphill $\perp$ the contours
10. power series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ (Desmos, converges for $|x| < 1$)
11. develop pattern for matching derivatives to build a polynomial

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for $x \in \mathbb{R}$ (MacLaurin series - centered at zero)
- $\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$ for $|x-1| < 1$ (Taylor series, radius of convergence 1)
- given "mystery function" with $f^{(n)}(3) = \frac{6n+2}{n^2+1}$, find $T_4(x)$

12. Quiz: find $T_4(x)$ if $f^{(n)}(8) = \frac{n^2+3}{2^n}$
13. When setting up an objective function, you may need to
    - substitute to enforce constraints
    - eliminate constants, squares, ln, exp, or other 1-1 functions
14. Find the point on the parabola $z = x^2 + y^2$ that is closest to $P(3, 1, 2)$.
    Let $Q(x, y, z)$ be constrained to the parabola. Let the objective function be the squared distance between P and Q: $f(x, y) = (x-3)^2 + (y-1)^2 + (x^2+y^2-2)^2$. The only critical point is a minimum.
15. Quiz: given Sage m.v. optimization output, classify the critical points using only the Hessian.

    $$z = (x-y)^2 * (y^2-1) + 5 * (x-2)^2 * (y-5) * x$$

    can do by inspection, without calculator (3 saddles, 1 l.max, 1 l.min)
16. In prison - to be released if you can use a simple calculator to find ln(20190211) to 6 decimal points.
    (a) you remember that $e \approx 2.72$, and $20190211/2.72^{17}$ is in the interval $(0, 2)$ for use with $\ln(x)$ Taylor series
    (b) for more precision, calculate $1/e = \sum \frac{(-1)^n}{n!} = 1 - 1 + 1/2 - 1/6 + 1/24 + \cdots \approx .36787944$
    (c) use the calculator to multiply by this number 17 times to get $x \approx .83586216$
    (d) $17 + \ln(x) = 17 + (x-1) - (x-1)^2/2 + (x-1)^3/3 - (x-1)^4/4 \cdots \approx 16.820708$
17. (see portfolio problem; motivate Lagrange mult)
    Use Desmos to plot $y = \frac{1}{2}x^2$ and $P(2, 5)$. What point Q on the parabola is closest to P ?
    (a) Solve using Calc I (critical pts and 2nd D.test) to find l.min, l.max, and l/g.min.
    (b) Illustrate that contours of $z = (x-2)^2 + (y-5)^2$ are concentric circles that are parallel to the constraint at the critical points.
    (c) Note that $\vec{PQ} = \begin{bmatrix} x-2 \\ \frac{1}{2}x^2 - 5 \end{bmatrix}$ is orthogonal to the parabola. (I'll be "right" there). The tangent vector to the parabola is $\vec{v} = \begin{bmatrix} 1 \\ \frac{dy}{dx} \end{bmatrix} = \begin{bmatrix} 1 \\ x \end{bmatrix}$, so solve $\vec{v} \cdot \vec{PQ} = 0$
    (d) The normal vectors to the objective function and constraint are:

    $$\nabla z = \begin{bmatrix} 2(x-2) \\ 2(y-5) \end{bmatrix}$$

    $$\nabla(\frac{1}{2}x^2 - y) = \begin{bmatrix} x \\ -1 \end{bmatrix}$$

    Set those to be parallel, so that $x - 2 = cx$, $y - 5 = -c$, and $y = \frac{1}{2}x^2$. Solve in W.A.
18. constrained optimization

(a) optimize $f(x, y)$ (the objective function) subject to $c(x, y) = 0$ (the constraint)

(b) the contours of f and c, and hence their gradients must be parallel, so the Lagrange equations are

$$\nabla f = \lambda \nabla c$$

$$c = 0$$

where λ is called the Lagrange multiplier.

(c) consistent and generalizable framework for solving on a computer, preferable to substitution methods

(d) solve the building heating cost problem using Lagrange

(e) biscuits and coffee - visualize the solution

19. A function that outputs a vector at every location (usually $\mathbb{R}^2$ or $\mathbb{R}^3$) is called a **vector field**. e.g. $F(x, y) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$ or $\begin{bmatrix} xy \\ x + y \end{bmatrix}$

20. corn field, hair scalp, wind velocity, force fields. Use software to plot vector fields along with the streamlines. A field with only unit vectors is $\frac{1}{\|F\|}F$, e.g. $\frac{1}{2(x^2+y^2)} \begin{bmatrix} x - y \\ x + y \end{bmatrix}$. If f is a scalar function, then ∇f is a vector field, with each gradient vector pointing uphill.

21. F is said to be "conservative" if there exists a "potential function" f with $\nabla f = F$. Under mild assumptions, the mixed partials must match, $F = \begin{bmatrix} P \\ Q \end{bmatrix}$ is conservative iff $P_y = Q_x$. To find the potential function, note $f = \int P dx + \boxed{\text{y stuff}} = \int Q dy + \boxed{\text{x stuff}}$; combine and check.

22. Move through a force field $F(x, y) = \begin{bmatrix} xy \\ x + y \end{bmatrix}$, along the "curve" $C : y = x^2$ from $(0, 0)$ to $(2, 4)$. The work done by F is

$$W = \int_C F \cdot dr = \int_C (xy)dx + \int_C (x + y)dy$$

Visualize this in Sage to see that generally the force field is pushing with us (doing positive work).

- parameterize C as $r(t) = (t, t^2)$ for $t \in [0, 2]$

$$W = \int_0^2 t^3(1dt) + \int_0^2 (t + t^2)(2tdt) = \int_0^2 2t^2 + 3t^3 dt = 52/3$$

- parameterize C as $r(t) = (e^t, e^{2t})$ for $t \in (-\infty, \ln(2)]$

$$W = \int_{-\infty}^{\ln(2)} e^{3t}(e^t dt) + \int_{-\infty}^{\ln(2)} (e^t + e^{2t})(2e^{2t}dt) = \int_{-\infty}^{\ln(2)} 3e^{4t} + 2e^{3t} dt = 52/3$$

23. Generally, if $F = \begin{bmatrix} P \\ Q \end{bmatrix}$, then the **line integral** for work is

$$\int_C F \cdot dr = \int_C \begin{bmatrix} P \\ Q \end{bmatrix} \cdot \begin{bmatrix} dx \\ dy \end{bmatrix} = \int_C P dx + \int_C Q dy = \int_{t_0}^{t_1} \left(P\frac{dx}{dt} + Q\frac{dy}{dt} \right) dt$$

It is not hard to prove that the choice of parameterization doesn't matter. Notationally, if we moved in the opposite direction, then $\int_{-C} F \cdot dr = -\int_C F \cdot dr$.

24. Go from $(0,0)$ to $(2,4)$ via $(t,2t)$, (t,t^2), $(t,t^3/2)$, $(t+\sin(\pi t), t^2+\sin(3\pi t))$. Do you always get the same answer ? If F is conservative (with $\nabla f = F$) then "work" corresponds to change in elevation (rise = slope $*$ run), so the integral is "path independent" (trail hiker vs mtn climber), and can be computed easily as the difference in potential: $\int_C F \cdot dr = f(b) - f(a)$. (FTC for line integrals). Consequently, if C makes a loop ending where it started (like the wise men), $b = a$, and $\int_C F \cdot dr = 0$.

25. With a slight change: $F(x,y) = \begin{bmatrix} xy \\ \frac{1}{2}x^2 + y \end{bmatrix}$ is conservative, and the potential function is $f(x,y) = \frac{1}{2}(x^2y + y^2)$ (plus arbitrary constant). Therefore if C goes from $(0,0)$ to $(2,4)$ by any route, then $\int_C F \cdot dr = f(2,4) - f(0,0) = 16$.

26. Another type of line integral is of a function with respect to arc length.

$$\int_C f(x,y)ds = \int_{t_0}^{t_1} f(x,y)\sqrt{(dx/dt)^2 + (dy/dt)^2}dt$$

If $f(x,y) = 1$, this just gives the length of the curve (assuming it is traced only once). Generally, the line integral could be thought of as the surface area of a wall with height f.

27. Of course, both types of line integrals can be expanded with more variables, e.g.

$$\int_C F \cdot dr = \int_C Pdx + \int_C Qdy + \int_C Rdz$$

$$\int_C f(x,y,z)ds = \int_{t_0}^{t_1} f(x,y,z)\sqrt{(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2}dt$$

28. Line integral practice. Here are somes curves:

- C_1: top half of unit circle traced counter-clockwise
- C_2: line from $(-1,0)$ to $(1,-4)$
- C_3: vertical line from $(1,-4)$ to $(1,0)$
- C_4: loop formed by $\vec{r}(t) = \begin{bmatrix} t(2-t) \\ t^2(2-t) \end{bmatrix}$
  note that this is a **positively oriented** (you travel around with the lake always on your left), **simple** (the curve doesn't cross itself), **closed** (it ends where it starts) curve
- C_5: path given by $x = 2t\cos(\frac{\pi}{9}t), y = 2^t, z = \frac{50t}{t^2+1}$ for $t \in [0,3]$.

Do each of these integrals as efficiently as possible.

(a) $\int_{C_1} x^2y ds$
Answer: parameterize and substitute

(b) $\int_{C_1 \cup C_2 \cup C_3} ds$
Answer: the perimeter $\int_{C_1} ds + \int_{C_2} ds + \int_{C_3} ds = \pi + \sqrt{20} + 4$

(c) $\int_{C_2} (y^2+2)dx + 2xydy$
Answer: use the potential function

(d) $\int_{C_1 \cup C_2 \cup C_3} (y^2+2)dx + 2xydy$
Answer: zero; $\int_C \nabla f \cdot dr = 0$

(e) $\int_{C_4} \frac{y}{x} ds$
Answer: substitute

(f) $\int_{C_4} \begin{bmatrix} y \\ x \end{bmatrix} \cdot dr$
Answer: conservative, so zero

(g) $\int_{C_4} \begin{bmatrix} x-y \\ x+y \end{bmatrix} \cdot dr$
Answer: not conservative, so substitute

(h) $\int_{C_5} \begin{bmatrix} z^2y+z \\ z^2x \\ 2xyz+x \end{bmatrix} \cdot dr$
Answer: use the potential function

29. Quiz: parameterize C by $x = \sin(\pi t), y = e^{-t^2}, t \in [0,3]$.

 (a) $\int_C y^2 x ds$

 (b) $\int_C \begin{bmatrix} y^2 \\ x \end{bmatrix} \cdot d\vec{r}$

30. Quiz: with C quarter-ellipse from $(4,e)$ to $(0,10)$ evaluate using the potential function:

$$\int_C 2x\ln(y)dx + \frac{x^2+y^2}{y}dy$$

Answer: $f(x,y) = x^2\ln(y) + \frac{y^2}{2}$, then $f(0,10) - f(4,e) = 50 - (16 + e^2/2) = 34 - e^2/2 \approx 30.3$

31. triple integral: work from inside-out, e.g.

$$\begin{aligned}
\int_1^3 \int_x^5 \int_0^{xy} ydzdydx &= \int_1^3 \int_x^5 (yz)_{z=0}^{z=xy} dydx \\
&= \int_1^3 \int_x^5 xy^2 dydx \\
&= \int_1^3 \frac{1}{3}(xy^3)_{y=x}^{y=5} dx \\
&= \frac{1}{3}\int_1^3 125x - x^4 dx \\
&= \left(\frac{125}{6}x^2 - \frac{1}{15}x^5\right)_{x=1}^{x=3}
\end{aligned}$$

32. Let R be a region bounded by $x = 2$ and $y = e^x$ in the 1st quadrant. Above R lies the surface $f(x,y) = 20 - xy$.

 (a) double integral: area $A = \iint_R 1dA$, and volume $V = \iint_R f(x,y)dA$

 (b) triple integral: volume, mass (say density is $\delta(x,y,z) = \frac{1}{1+z^2}$)

$$V = \int_0^2 \int_0^{e^x} (20-xy)dydx = \int_0^2 \int_0^{e^x} \int_0^{20-xy} 1dzdydx$$

$$M = \int_0^2 \int_0^{e^x} \int_0^{20-xy} \delta(x,y,z)dzdydx$$

 (c) "moments", e.g. M_{xz}, then integrate y since that's the distance to the xz plane.

 (d) center of mass $\left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$; "centroid" uses constant density

 (e) avg dist to origin

  - by volume $\frac{\iiint_S \sqrt{x^2+y^2+z^2}dV}{V}$

- weighted by density $\dfrac{\iiint_S \delta(x,y,z)\sqrt{x^2+y^2+z^2}dV}{M}$

33. Let S be top hemisphere of $x^2+y^2+z^2 = r^2+z^2 = 1$.
    (a) triple integral for volume $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} 1rdzdrd\theta$
    (b) centroid $(0,0,.375)$
    (c) center of mass if density proportional to dist. to origin $(0,0,.4)$
    (d) center of mass if density inversely proportional to dist. to origin $(0,0,1/3)$

34. Let R be bounded by $y = 1-x$ in 1st quadrant. Let L be the diagonal line segment. We will pick points "randomly" from R (uniformly w.r.t. area) and L (uniform w.r.t. length)
    (a) average squared distance from pt $(t, 1-t)$ on L to the origin. note that $ds = \sqrt{1^2+(-1)^2} = \sqrt{2}$ (the denom is the length of L).

    $$\frac{\int_L (t^2+(1-t)^2)ds}{\int_L 1ds} = \frac{\int_0^1 (t^2+(1-t)^2)\sqrt{2}dt}{\int_0^1 \sqrt{2}dt}$$

    note that avg distance $\neq$ square root of the avg squared distance
    (b) average squared distance from pt (x,y) in R to the origin (the denom is the area of R)

    $$\frac{\iint_R (x^2+y^2)dA}{\iint_R 1dA} = \frac{\int_0^1\int_0^{1-x}(x^2+y^2)dydx}{\int_0^1\int_0^{1-x}1dydx}$$

    (c) avg squared dist from pt (x,y) in R to pt $(t, 1-t)$ on L (denom is area of R times length of L)

    $$\frac{\iint_R\int_L((x-t)^2+(y-1+t)^2)dsdA}{\iint_R\int_L 1dsdA} = \frac{\int_0^1\int_0^{1-x}\int_0^1((x-t)^2+(y-1+t)^2)\sqrt{2}dtdydx}{\int_0^1\int_0^{1-x}\int_0^1 1\sqrt{2}dtdydx}$$

    (d) avg squared dist between two points (x,y) and (v,w) in R (quadruple integral)

    $$\frac{\int_0^1\int_0^{1-x}\int_0^1\int_0^{1-v}((x-v)^2+(y-w)^2)dwdvdydx}{.25} = \frac{4}{18}$$

35. Quiz: evaluate $\displaystyle\int_{-1}^1\int_1^{1+\theta}\int_0^{r\theta} rdzdrd\theta$, by hand showing all steps.
    Answer: 4/5

36. Initial Value Problems: position $\vec{r}$, velocity $\vec{v} = \frac{d\vec{r}}{dt}$, acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$. Integrate to work backwards, applying initial conditions, e.g.

    $$\vec{v}(t) = \begin{bmatrix} e^{-2t} \\ t^3 \end{bmatrix} \qquad \vec{r}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

    Note: starts at $(4,1)$ heading east, then turns north. Exact position at any other time (future or past):

    $$\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{v}(\tau)d\tau = \begin{bmatrix} 4.5 - \frac{1}{2}e^{-2t} \\ 1 + \frac{1}{4}t^4 \end{bmatrix}$$

    CHECK your answer satisfies given conditions.

    Alternatively, you can use an indefinite integral and solve for constants:

    $$\vec{r}(t) = \int \vec{v}(t)dt + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

37. Force acts to accelerate $\vec{a}(t) = [\sin(t), 12t^2, 60t^3]$. If $\vec{v}(0) = [0, 0, 4]$ and $\vec{r}(1) = [3, 2, 5]$, find $\vec{r}(0)$.

$$\vec{v}(t) = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} -\cos(t) \\ 4t^3 \\ 15t^4 \end{bmatrix} - \begin{bmatrix} -\cos(0) \\ 4(0)^3 \\ 15(0)^4 \end{bmatrix} = \begin{bmatrix} 1-\cos(t) \\ 4t^3 \\ 4+15t^4 \end{bmatrix}$$

$$\vec{r}(t) = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} t-\sin(t) \\ t^4 \\ 4t+3t^5 \end{bmatrix} - \begin{bmatrix} 1-\sin(1) \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 2+\sin(1)+t-\sin(t) \\ 1+t^4 \\ -2+4t+3t^5 \end{bmatrix}$$

so $\vec{r}(0) = [2+\sin(1), 1, -2]$.

38. Quiz: if $\vec{a}(t) = \begin{bmatrix} 2t \\ \cos(\pi t) \end{bmatrix}$, $\vec{v}(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, and $\vec{r}(2) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, then find $\vec{r}(t)$.

Answer: $\vec{r}(t) = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}(t-2) + \begin{bmatrix} t^3/3 - 8/3 \\ \frac{1}{\pi^2} - \frac{1}{\pi^2}\cos(\pi t) \end{bmatrix} = \begin{bmatrix} -23/3+4t+t^3/3 \\ 3+\frac{1}{\pi^2}+t-\frac{1}{\pi^2}\cos(\pi t) \end{bmatrix}$

39. Geogebra illustration of radius of curvature. Acceleration is change in speed or direction. Consider driving around a circle $\vec{r}(t) = \begin{bmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{bmatrix}$.

- Compute $\vec{v}(t)$ and show $\vec{v} \perp \vec{r}$.
- Compute $\vec{a}(t)$ and show $\vec{a} = -\omega^2\vec{r}$. (centrepetal force to keep you on the circle)
- Note that speed is constant $\|\vec{v}\| = |\omega R|$.
- Magnitude of acceleration is $\|\vec{a}\| = \omega^2 R$ (if you double speed, then acceleration quadruples)

40. In general, set up a tangent, normal coordinate system with unit vectors and decompose

$$\vec{a} = a_T\frac{\vec{v}}{\|\vec{v}\|} + a_N\vec{n}$$

- $a_T = \|\vec{a}\|\cos(\theta) = \frac{\vec{a}\cdot\vec{v}}{\|\vec{v}\|}$
- $a_N = \|\vec{a}\|\sin(\theta) = \frac{|\vec{a}\times\vec{v}|}{\|\vec{v}\|}$

In the case of our constant speed circle, $a_N = \|\vec{a}\| = \omega^2 R = \frac{\|v\|^2}{R}$, so generalize the radius of curvature to be

$$R = \frac{\vec{v}\cdot\vec{v}}{a_N} = \frac{\|\vec{v}\|^3}{|\vec{a}\times\vec{v}|}$$

And define the **curvature** to be $\kappa = \frac{1}{R}$.

This seems right because if you double speed, acceleration quadruples, so that R, κ stay constant.

41. In the case $y = f(x)$, then parameterize $(t, f(t))$ so $\vec{v}(t) = (1, f'(t))$ and $\vec{a}(t) = (0, f''(t))$. Then curvature is:

$$\kappa(x) = \frac{|\vec{a}\times\vec{v}|}{\|\vec{v}\|^3} = \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

42. At time $t = 3$ you know $\vec{r}(3) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v}(3) = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\vec{a}(3) = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$.

(a) tangent line $\vec{\ell}(t) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}(t-3)$

(b) speed $\|\vec{v}(3)\| = \sqrt{38} \approx 6.1644$

(c) find a_T, a_N, κ, R using formulas

(d) approximate speed at $t = 3.05$ by linearizing $\Delta\vec{v} \approx \frac{d\vec{v}}{dt}\Delta t$
$\vec{v}(3.05) \approx \vec{v}(3) + \vec{a}(3)(t-3) = [2.05, 3.2, 5.45]$
so $\|\vec{v}(3.05)\| \approx 6.6442$
NOTE: speeding up since $\frac{d}{dt}\|\vec{v}\| = a_T > 0$

43. The **del** (**nabla**) operator is formally $\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$.

Let f be a scalar function, and $\vec{F} = [P, Q, R]$ be a vector field.

- **gradient** $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right] \in \mathbb{R}^3$
  describes the slope of the surface
- **divergence** $\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \in \mathbb{R}$
  describes the net flux at that point (positive is a source, negative is a sink)
- **Laplacian** $\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \in \mathbb{R}$
  describes the average nearby value of f (positive is higher, negative is lower)
- **curl** $\nabla \times \vec{F} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ P_z - R_x \\ Q_x - P_y \end{bmatrix} \in \mathbb{R}^3$
  describes the rotation (axis and magnitude) around that point