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# MATH 211 Test 3, Fall 2018

**Directions:**

- You may use a calculator for arithmetic calculations.
- This exam is open book/notes. But do not use the internet.
- You have 60 minutes. You must work alone; do not communicate with any other person.
- To receive full credit, you must **show all relevant work to completely justify your answer (on separate paper)**.
- Use notation conventions from class.
- 55 points possible, graded out of 50 points.

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1. (5 pts) At a certain point on the graph of a function $f(x,y)$, the steepest descent direction is $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, having a slope of $-0.43$. Find the gradient $\nabla f$ at that point.
   **Answer:** $\nabla f = \frac{-.43}{\sqrt{29}} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

2. (5 pts) Let $z = \frac{x^2+3}{(3y+1)^2}$. Find $\frac{\partial^2 z}{\partial x \partial y}$.
   **Answer:** $\frac{\partial z}{\partial x} = \frac{2x}{(3y+1)^2}$, adn $\frac{\partial^2 z}{\partial x \partial y} = -12x(3y+1)^{-3}$

3. (5 pts) Consider moving along the surface of $z = f(x,y)$ such that $x = 9 + t^2$ and $y = \frac{1}{t}$.
   At the point where $t = 2$, if $\frac{\partial f}{\partial y} = 5$ and $\frac{dz}{dt} = 8$, then find $\nabla f$.
   **Answer:** by chain rule: $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$, so $8 = (f_x)(2t) + (5)(-1/t^2)$, and when $t = 2$, $8 = 4f_x - 5/4$, so $f_x = 2.3125$ and $\nabla f = [2.3125, 5]$

4. (5 pts) Evaluate this improper integral:
   $$\int_0^\infty \left( \int_1^3 y e^{-x/5} dy \right) dx$$
   **Answer:** $\int_0^\infty 4e^{-x/5} dx = 20$

5. (7 pts) Suppose $z$ is a function of $x$ and $y$. At point $P(5, 8, 37)$ on the surface you know
   $dz = 0.34dx + 0.81dy$.
   (a) If you turn the knobs $\Delta x = 0.15$ and $\Delta y = -0.04$, then estimate the change in output $\Delta z$.
   **Answer:** $(.34)(.15) + (.81)(-.04) = .0186$
   (b) Find the equation of the line through $P$ normal to the surface.
   **Answer:** $x = 5 + .34t, y = 8 + .81t, z = 37 - t$

6. (7 pts) Let $f(x,y) = \ln(9 - |x| - |y|)$.
   (a) Find the range of $f$.
   **Answer:** $(-\infty, \ln(9)]$

(b) Sketch the domain of $f$, along with the contours $f(x, y) = -2, -1, 0, 1, 2$.
Your sketch should clearly show which of those contours are closer together.
**Answer:** concentric diamonds, closer together for smaller values of $f$

7. (10 pts) Suppose that at a certain point $\nabla f = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ and the Hessian matrix is $H = \begin{bmatrix} 9 & 5 \\ 5 & -3 \end{bmatrix}$.
You move in the direction $\vec{v} = \begin{bmatrix} 1 \\ b \end{bmatrix}$.

(a) If $b = 2$, find the slope (directional derivative).
**Answer:** $\frac{(7)(1)+3(2)}{\sqrt{1^2+2^2}} = 13/\sqrt{5}$

(b) If $b = 2$, find the concavity.
**Answer:** $\frac{\vec{v}\cdot(H\vec{v})}{\vec{v}\cdot\vec{v}} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} (9)(1) + (5)(2) \\ (5)(1) + (-3)(2) \end{bmatrix}}{5} = 17/5$

(c) Find a value of $b$ so that the slope equals exactly 1.
**Answer:** solve $\frac{7+3b}{\sqrt{1+b^2}} = 1$ to get (by QF) $b = -3.57, -1.68$, but only $b = -1.68$ satisfies the equation.

8. (11 pts) A rectangular city is bounded by $|x - 2| = 5$ and $|y - 4| = 7$.
There is a Dollar General at $G(0, 0)$, and a Dollar Tree at $T(1, 8)$.

(a) Sketch the city's region, $R$, and label the dollar store locations.

(b) Let $P(x, y)$ be a point in the city. Find the average (over all such points in the city) perimeter of $\Delta PGT$. Set up the calculation, but do not evaluate the double integral.
**Answer:** the city's area is 140, and the perimeter of the triangle is the sum of the legs, so the average perimeter is $\dfrac{\int_{-3}^{11}\int_{-3}^{7}(\sqrt{x^2+y^2} + \sqrt{65} + \sqrt{(x-1)^2+(y-8)^2})dxdy}{140}$

(c) The population density of this city is $\delta(x, y) = 1 + 600xy(100 + x^2 + y^2)^{-2}$ thousand people per square km. Find the city's total population by computing $\iint\limits_R \delta(x, y)dA$

**Answer:** $140 + \int_{-3}^{11}\int_{-3}^{7}\delta(x, y)dxdy = \int_{-3}^{11} -300y(149 + y^2)^{-1} + 300y(109 + y^2)^{-1}dy = 140 + 150(\ln(230) - \ln(270) + \ln(158) - \ln(118)) = 159.73$