NAME ____________________________

MATH 211 Test 4, Fall 2019

Directions:

- This exam is open book/notes. But do not use the internet.
- You may use a calculator for arithmetic calculations.
- You have 55 minutes. You must work alone; do not communicate with any other person.
- To receive full credit, you must **show all relevant work to completely justify your answer (on separate paper)**.
- 105 points possible, graded out of 100 points.

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1. (30 pts) Let $f(x, y)$ be a function with $\nabla f = \begin{bmatrix} y^2 - 8y + 8x \\ 2xy - 8x \end{bmatrix}$. Find the critical points and classify each one using the 2nd derivative test.
   Answer: set $2x(y-4) = 0$ to get $x = 0$ or $y = 4$, then get c.pts $(0, 0)$, $(0, 8)$, and $(2, 4)$.
   $H = \begin{bmatrix} 8 & 2y - 8 \\ 2y - 8 & 2x \end{bmatrix}$
   at $(0, 0)$, $D < 0$, so saddle
   at $(0, 8)$, $D < 0$, so saddle
   at $(2, 4)$, $D > 0$ and $f_{xx} > 0$, so CU, local min

2. (15 pts) Consider the surface of $z = f(x, y)$, and consider a path across that surface with:

$$x = 4t - 3, \qquad y = \frac{15}{1 + t^2}$$

   At the point where $t = 2$, suppose $\frac{\partial z}{\partial y} = -2$ and $\frac{dz}{dt} = 10$. Write the chain rule expression for $\frac{dz}{dt}$, and use it to to find $\frac{\partial z}{\partial x}$ at the given point.
   Answer: by chain rule: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$, so $10 = (f_x)(4) + (-2)(-60/25)$, and $\frac{\partial z}{\partial x} = 1.3$

3. (35 pts) A quadratic function $f(x, y)$ satisfies these conditions:

   - $f(0, 0) = 50$
   - $\nabla f(0, 0) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$
   - $H = \begin{bmatrix} 6 & 7 \\ 7 & -4 \end{bmatrix}$

   (a) Find the formula for $f(x, y)$.
   Answer: $f(x, y) = 50 + 2x + 5y + 3x^2 - 2y^2 + 7xy$

   (b) Find the (x, y, z) coordinates of the critical point. Is it a local max, local min, or saddle?
   Answer: solve $2 + 6x + 7y = 0, 5 - 4y + 7x = 0$ to get $x = -43/73, y = 16/73, z \approx 49.6$

   (c) Find the concavity in the direction of $\vec{v} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$.
   Answer: $\frac{\vec{v} \cdot (H\vec{v})}{\vec{v} \cdot \vec{v}} = \frac{1}{85} \begin{bmatrix} 9 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 68 \\ 55 \end{bmatrix} = 722/85$

   (d) The concavity in the $\vec{w} = \begin{bmatrix} 1 \\ a \end{bmatrix}$ direction is zero. Find the value of $a > 0$.
   Answer: concavity works out to $6 + 14a - 4a^2 = 0$, by the QF $a \approx 3.886$

4. (25 pts) Suppse z is an implicit function of x and y such that

$$z^2 x + 10 = y^2 + z$$

(a) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
Answer: $2z\frac{\partial z}{\partial x}x + z^2 = \frac{\partial z}{\partial x}$, so $\frac{\partial z}{\partial x} = \frac{z^2}{1-2xz}$
$2z\frac{\partial z}{\partial y}x = 2y + \frac{\partial z}{\partial y}$, so $\frac{\partial z}{\partial y} = \frac{2y}{2xz-1}$

(b) Find the equation of the tangent plane at the point where $y = 5$ and $z = 3$
Answer: solve to get $x = 2$; the gradient is $\nabla z = \begin{bmatrix} -9/11 \\ 10/11 \end{bmatrix}$,
so the tan.plane is $z = 3 - \frac{9}{11}(x-2) + \frac{10}{11}(y-5)$

(c) At that point, find dz if $dx = .40$ and $dy = .15$.
Answer: $-9/11(.4) + 10/11(.15) = -.191$